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CONTENTS

Chemistry. — *Die akute Zinnpest* III. Von ERNST COHEN und W. A. T. COHEN–DE MEESTER.

(Communicated at the meeting of September 24, 1938.)

Einleitung.

In unseren beiden früheren Abhandlungen über die akute Zinnpest [1]) erörterten wir die Art und Weise in der sich dieselbe durch Zusatz äusserst geringer Spuren (z.B. 0.001 bezw. 0.01 Gew. Proz.) Aluminium zum weissen Zinn hervorrufen lässt. Dabei ergab sich, dass der in erster Instanz eintretende HEYN–WETZELeffekt (H.W.E.) diese merkwürdige Erscheinung erzeugt. Es erhebt sich nunmehr die Frage, ob die akute Zinnpest auch durch den Zusatz minimaler Spuren von anderen Metallen eintritt und ob auch dann der H.W.E. zur Erklärung herangezogen werden kann.

Die vorliegende Abhandlung befasst sich in erster Linie mit der Beantwortung dieser Frage, ausserdem aber mit dem für die Praxis nicht weniger interessanten Problem: wie lässt sich die enorme Verzögerung der U.G. des weissen Zinns in die graue Modifikation erklären, welche man beim Zusatz minimaler Spuren von Bi, Sb, Pb zum weissen Zinn beobachtet, eine Erscheinung, von der bereits in unserer ersten Abhandlung die Rede war.

A. *Welche Metalle erzeugen die akute Zinnpest?*

1. Wie in den §§ 3, 4 und 5 unserer zweiten Mitteilung nachgewiesen wurde, tritt der H.W.E. beim weissen Zinn, welches Spuren Al enthält, dadurch ein, dass die Sn-Al Legierung die Spuren Wasser, welche in jedem Metall vorhanden sind, zersetzt. Es liegt somit auf der Hand zu vermuten, dass der Zusatz von Spuren solcher Metalle zum weissen Zinn, welche damit eine das Wasser zersetzende Legierung bilden, zur Erzeugung der akuten Zinnpest führen.

2. Sucht man einen Leitfaden, welcher sich zum ausfindig machen von derartigen Legierungen eignet, so stösst man auf gewisse Schwierigkeiten, welche sich aus folgender Ueberlegung ergeben. Handelt es sich bei *reinen* Metallen darum solche Legierungen aufzufinden, welche das Wasser zersetzen, so gibt *grosso modo* die Betrachtung der elektroche-

[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **40**, 746 (1937); **41**, 462 (1938). Auch Z. physik. Chem. (A) **181**, 124 (1937); **182**, 103 (1938).

mischen Spannungsreihe K, Na, Ba, Sr, Ca, Mg, Al, Mn, Zn, Fe, Ni, Co, Cd, Tl, Sn, Pb, H_2, Cu, Bi, Sb, Hg, Ag, Pd, Pt, Au darüber Aufschluss. Bekanntlich zerlegen die dem Wasserstoff vorangehenden Metalle das Wasser.

Nun fusst diese Reihe auf Potentialmessungen, bei welchen die betreffenden Metalle sich in den 1-N Lösungen ihrer Salze befinden. Diese Messungen sind nun aber bisher in den meisten Fällen sehr unsicher. Manche Metalle überziehen sich in Berührung mit Wasser mit einer schützenden Haut von Hydroxyd, welche die Potentialmessungen fälscht, andere werden passiv, wieder andere zeigen eine „Ueberspannung" (overvoltage). Bei diesen Erscheinungen spielen u.a. die Oberflächenbeschaffenheit des Metalls, dessen thermische Vorgeschichte, geringe Verunreinigungen (auch Gase wie H_2 bezw. O_2) und besonders auch die Hydrolyse der verwendeten Lösungen, welche die Ionenkonzentration derselben unbestimmt macht, eine wichtige Rolle. Infolgedessen kann sich von Fall zu Fall die relative Stellung des Wasserstoffs und der Metalle in der Spannungsreihe ganz bedeutend ändern. Damit verliert dieselbe aber ihre allgemeine Bedeutung für das Auffinden wasserzersetzender Metalle.

3. In unserem Falle, wo es sich um das Auffinden von wasserzersetzenden Zinn-*legierungen* handelt, ist die Sachlage eine weit kompliziertere, da uns bisher eine scharf definierte elektrochemische Spannungsreihe der Legierungen in ihrer Abhängigkeit von deren Zusammensetzung noch gänzlich fehlt. Auch über den Einfluss, den der Zusatz von Spuren von Fremdmetallen auf die Passivität eines bestimmten Metalles ausübt, ist bisher Näheres nicht bekannt geworden. Es ist somit auch nach dieser Richtung vorderhand nicht möglich eine Prognose über das Verhalten einer Legierung dem Wasser gegenüber zu machen.

Es bleibt uns demnach vorläufig nichts anderes übrig, als durch den Versuch festzustellen, ob eine gegebene Zinnlegierung das Wasser zersetzt, d.h. ob dieselbe in Berührung mit Wasser Wasserstoff entwickelt. Ist dies der Fall, so kann der H.W.E. eintreten, und damit auch die akute Zinnpest, wenn man die betreffende Legierung Temperaturen unterhalb des Umwandlungspunktes des Zinns (+ 13.2° C.) aussetzt.

B. *Die Versuchstechnik.*

4. Wir verwendeten das 99.996 Gew. Proz. Zinn von HILGER in London (H.Z. vergl. § 4 unserer ersten Mitteilung). Die diesem in Mengen von 0.01 bezw. 0.05 Gew. Proz. zugesetzten reinen Metalle, von derselben Firma bezogen, waren ebenfalls spektrografisch analysiert worden. Ihre Reinheitsgrade waren: Mg 99.82; Zn 99.999^5; Ni 99.97; Cd 99.999^0 Gew. Proz. Das verwendete Tl war von uns auf elektrolytischem Wege aus mehrfach umkristallisiertem Thallosulfat hergestellt worden. Das Fe war Ferrum reductum pro analysi.

Wir erhitzten dieses vor dem Gebrauch in einem Wasserstoffstrom, der zuvor nach dem von ERNST COHEN und H. R. BRUINS beschriebenen Verfahren sorgfältigst gereinigt war [1].

Die Legierungen stellten wir im Hochvakuum dar durch Zusammenschmelzen von gewogenen Mengen der betreffenden Metalle in Pyrexkugeln, an welche ein etwa 7 mm weites Pyrexrohr angeschmolzen war, das nach dem Evakuieren vor der Lampe geschlossen wurde. Nach dem Schmelzen der Legierung dreht man das Ganze um: die flüssige Legierung sammelt sich in dem Rohr und bildet nach dem Erkalten an der Luft einen Metallstab. Nach dem Entfernen des Glasrohrs walzten wir diesen Stab bei Zimmertemperatur zu Folie von 15 mm Breite und 0.2 mm Dicke herunter.

C. *Die Ergebnisse.*

5. Ueber die Ergebnisse können wir uns kurz fassen: Nur die Mg-Legierung (0.01 Gew. Proz. Mg.) zeigte in Berührung mit Wasser den H.W.E. (wie Fig. 1 in natürlicher Grösse dartut) und dementsprechend bei — 50° C. (im zugeschmolzenen Glasrohr) die akute Zinnpest. Als Vergleichsobjekt wurde in jedem Versuch reines H.Z. verwendet, welches dieselbe thermische und mechanische Vorbehandlung wie die Legierungen durchgemacht hatte.

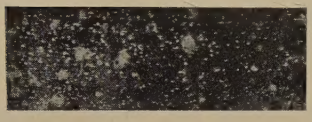

Fig. 1.

Wir haben bereits früher (Erste Mitteilung § 5) darauf hingewiesen, dass die beschriebene Erscheinung sich auch bei anderen Metallen zum Studium ihrer polymorphen Umwandlungen heranziehen lässt. Wir hoffen darüber demnächst zu berichten und wenden uns nunmehr der Frage zu:

D. *Wie erklärt sich die enorme Verzögerung der U. G. des weissen Zinns in das graue beim Zusatz minimaler Spuren gewisser Metalle?*

6. In ihren Arbeiten über den Einfluss mechanischer Deformation auf die U.G. polymorpher Metalle wurde bereits von ERNST COHEN und A. K. W. A. VAN LIESHOUT [2] darauf hingewiesen, dass J. FRITZSCHE [3], P. FARUP [4] sowie auch G. TAMMANN und K. L. DREYER [5] den verzögernden Einfluss geringer metallischer Beimengungen auf die U.G.

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[1] Z. physik. Chem. **94**, 443 (1920).

[2] Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 352 (1936). Auch Z. physik. Chem. (A) **177**, 331 (1936).

[3] Mém. Acad. Imp. Sci. Pétersbourg (7) **15**, No. 5 (1870).

[4] Teknisk Ugeblad **55**, 29. Mai (1908); Tids. Kemi. Farm. Terapi No. 11 und 12 (1908).

[5] Z. anorg. allgem. Chem. **199**, 97 (1931).

des weissen Zinns in die graue Modifikation studiert haben. Da diesen Forschern jedoch damals der enorme Einfluss der thermischen und mechanischen Vorgeschichte der untersuchten Objekte, sowie derjenige der Gegenwart von SnO_2 auf die U.G. des weissen Zinns, welche Einflüsse erst später von ERNST COHEN und A. K. W. A. VAN LIESHOUT [1]) festgestellt wurden, nicht bekannt war, m.a.W.: da die von ihnen untersuchten Objekte nicht genügend definiert waren, konnten die von ihnen beobachteten Erscheinungen für uns nicht zum Ausgangspunkt gewählt werden.

7. Anders liegt die Sache bei den Studien von C. W. MASON und W. D. FORGENG [2]), welche ein scharf definiertes Material verwendeten. Es ergab sich, dass die U.G. des spektrographisch reinen, weissen Zinns durch den Zusatz minimaler Spuren von Bi ganz enorm herabgesetzt werden kann, wobei die thermische Vorgeschichte des untersuchten Materials bestimmend wirkt.

Da ihre überaus sorgfältige Arbeit den Weg zeigt, welcher auch zur Erklärung in letzter Instanz der verzögernden Wirkung anderer Metalle zu führen im Stande sein dürfte, seien hier die wichtigsten Ergebnisse der amerikanischen Forscher kurz erörtert. Für die freundliche Ueberlassung ihrer Mikrophotogramme, von welchen hier mehrere reproduziert sind, bringen wir ihnen auch an dieser Stelle unseren besten Dank.

8. Vorausgeschickt sei, dass der Erfolg ihrer Untersuchung der Verwendung eines speziell von ihnen ausgearbeiteten Aetzverfahrens zu verdanken ist, welches sie in stand setzte die Strukturänderungen, welche in den studierten Zinnobjekten bei der Kristallisation eintreten, auf mikrophotographischem Wege zu verfolgen. Diese Aenderungen treten nicht zu Tage bei Anwendung der bisher üblichen Aetzmethoden. Da ihr Verfahren in Zukunft beim Studium ähnlicher Probleme von grosser Bedeutung werden dürfte, und nur das genaue Befolgen ihrer Vorschriften zum Ziel führt, sei zunächst die Beschreibung derselben hier wiedergegeben:

„Die übliche Bearbeitung der Oberfläche des zu untersuchenden Objekts findet mittels Schmirgelpapiers No. 000 statt. Sodann bringt man es auf eine mit Baumwollstoff bespannte Polierscheibe, welche wenigstens 400 Umdrehungen pro Minute macht. Die Baumwolle wird mit nassem Carborundumpulver No. 600 bestreut. Zunächst soll die Richtung des Schleifens die der Striche, welche von dem zuletzt verwendeten Schmirgelpapier herrühren, unter einem Winkel von 90° schneiden. Sind letztere entfernt, so führt man das Objekt langsam über die Scheibe hin und her.

Die nächste Stufe des Polierens wird auf einer Scheibe ausgeführt, welche ein so fein zerteiltes Pulver von Schmirgel oder Aluminiumoxyd

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1) Z. physik. Chem. (A) **173**, 1 (1935) Speziell daselbst § 34.

2) Metals and Alloys, April 1935.

trägt, dass sich dasselbe in Wasser um nicht mehr als 15 cm in 3 Std. senkt.

Nachdem die letzten Kratzspuren, welche von dem Schmirgel No. 600 herrühren, entfernt sind, wäscht man das Objekt mit Wasser und Alkohol und trocknet es in einem Luftgebläse. Das Aetzen erfolgt, indem man das Metall sanft mit einem Baumwollebausch betupft, welcher mit einer 5-prozentigen Salpetersäurelösung in *absolutem* Alkohol durchtränkt ist. Die Aetzung soll so tief sein, dass sie jedwede „geflossene" Oberflächenschicht entfernt. Sodann wird das Objekt nochmals auf der Scheibe poliert und leicht angeätzt. Gewöhnlich genügen diese Manupulationen zum Sichtbarmachen der Struktur, aber in manchen Fällen, speziell wenn es sich um sehr feine Strukturen handelt, dürfte es erforderlich sein, ein leichtes Polieren und Aetzen zu wiederholen. Die Verwendung von *absolutem* Alkohol ist unbedingt notwendig. Ist Wasser zugegen, wie es im gewöhnlichen 95-prozentigen Alkohol der Fall ist, oder wird ein Objekt geätzt, wenn es noch nass ist, so treten nur die Umrisse von Körnern und orientierter Glanz zu Tage. Die Strukturen, welche das später zu erörternde, anomale Verhalten des Objektes erklären, bleiben dann völlig verdeckt. Die Aetzmittel, welche gewöhnlich für Zinn verwendet werden, sind nach dieser Richtung sämtlich unbefriedigend, obwohl die Befeuchtung mit einer 5-prozentigen Silbernitratlösung oder das Aetzen auf elektrolytischem Wege, die betreffenden Strukturen mehr oder weniger ausgesprochen hervortreten lässt".

9. Wurde das von MASON und FORGENG zunächst verwendete Zinn (Verunreinigung Bi 0.0035 Gew. Proz.; Cu 0.0017; Pb 0.0035; Fe 0.0017; Sb 0.0018; S, As, Zn liessen sich nicht nachweisen. Analyse des Bureau of Standards in Washington D.C.) nach dem Schmelzen und darauf folgendem Abschrecken auf Zimmertemperatur in einer Graphitform, nach dem beschriebenen Verfahren geätzt, so zeigte es die in Fig. 2 abgebildete Struktur (Vergrösserung 50 ×). Temperte man es sodann während 50 Std. bei 200° C., so verschwand dieselbe und die Legierung wurde homogen (Fig. 3, Vergrösserung 50 ×).

Die Verunreinigungen, welche in die dunklen Teile (Fig. 2) segregiert waren, sind offenbar in homogene feste Lösung getreten. Wird ein solches Objekt örtlich (Ecke rechts) zum Schmelzen gebracht und dann abgeschreckt, so tritt, wie Fig. 4 (Vergrösserung 50 ×) zeigt, die Duplexstruktur wieder auf. Es stellte sich nun heraus, dass die abgeschreckten Objekte bei —40° C. bereits nach 24 Std. von der Zinnpest befallen waren, die getemperten dagegen selbst nicht nach sechs Monaten.

10. Stellte man nach dem von MASON und FORGENG ausgearbeiteten Elektrolysenverfahren (Vergl. unsere erste Mitteilung[1]) § 1) ein Zinn-

[1]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **40**, 746 (1937); Z. physik. Chem. (A) **181**, 124 (1937).

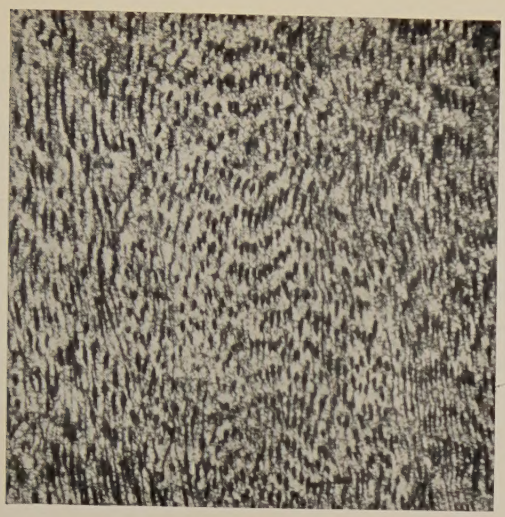

Fig. 2.

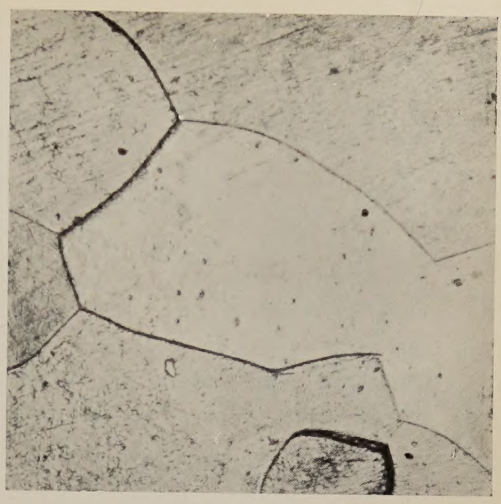

Fig. 3.

Fig. 4.

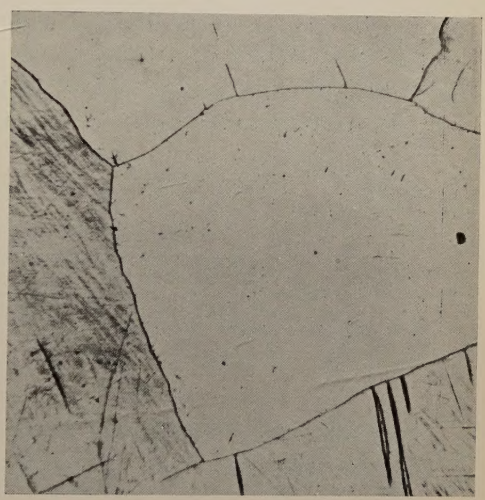

Fig. 5.

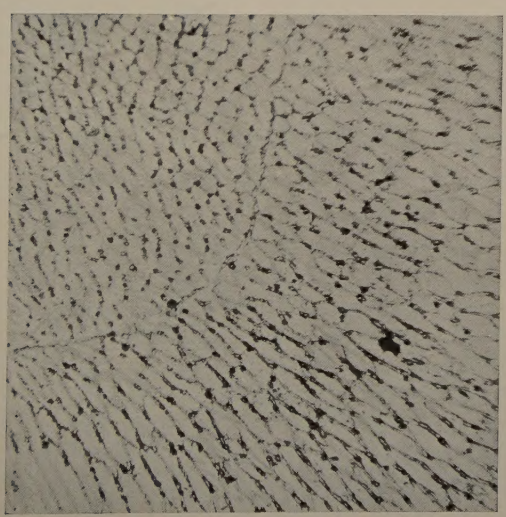

Fig. 6.

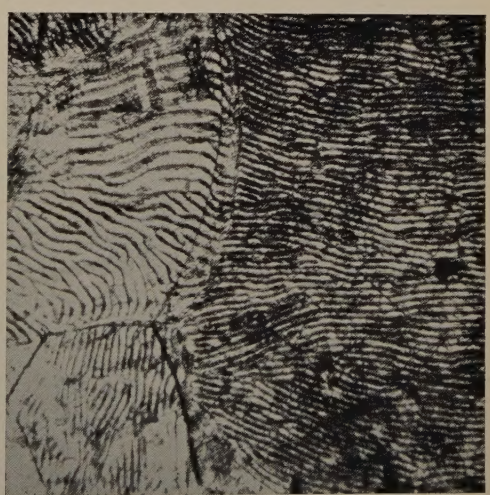

Fig. 7.

präparat her, welches, wie die spektrographische Analyse ergab, völlig Bi-frei war, so trat bei diesem weder nach starkem Abschrecken, noch nach langsamem Abkühlen, beim Aetzen irgend welche Struktur auf (Fig. 5, Vergrösserung 100 ×) [1].

Setzt man diesem, durch Elektrolyse gereinigten, Zinn Spuren Bi zu (z.B. 0.001 Gew. Proz.), so tritt beim Abschrecken von 300 auf 25° C. und nachfolgendem Aetzen die Struktur wieder auf (Fig. 6, Vergrösserung 50 ×), während ein 0.004 Gew. Proz. Bi enthaltendes Präparat nach dieser Vorbehandlung die in Fig. 7 (Vergrösserung 100 ×) abgebildete Struktur zeigt. Wird dieses Präparat getempert, so kommt die Struktur zum Verschwinden und zeigt wieder das in Fig. 3 dargestellte Bild.

11. Die Autoren weisen darauf hin, dass die beschriebenen Strukturen nicht Gleichgewichte darstellen; selbst die langsam abgekühlten Objekte sind nicht stabil. Sie enthalten weit weniger Bi als der Löslichkeit desselben in fester Lösung im Zinn (etwa 1 % bei Zimmertemperatur) entspricht. Die beobachteten Strukturen sind eine Folge der Kernbildung („coring") der Zinnkristalle, wenn diese sich aus der Schmelze bilden.

Die Lage der dunkeln Gebilde von Bi-reichen Teilen entspricht den Interstitien eines dendritischen Wachstums; die Feinheit der Gebilde ändert sich mit der Bildungsgeschwindigkeit der Dendriten. Zinn zeigt eine ausserordentlich starke Neigung zur Dendritenbildung.

Die Kernbildung ist beim Zinn ausserordentlich stark; der Nachweis, dass sie stattfindet, gelingt indes nur bei Verwendung eines geeigneten Aetzmittels und es ist wahrscheinlich, dass Fälle dieser Art auch bei anderen Legierungen vorliegen.

Selbstverständlich sind die primär gebildeten Kanten der Zinndendriten um so reiner, je stärker die Kernbildung stattfindet. Daher ist es verständlich, dass die allotrope Umwandlung des weissen Zinns in die graue Modifikation, in der getemperten und homogenen festen Lösung, welche einige Tausendstel Prozente Bi enthält, nicht stattfindet, sondern, dem praktisch reinen Zinn entlang, im Innern der Kristallkerne verläuft.

Dieses Verhalten beobachteten die Autoren tatsächlich im Mikroskop: die allotrope Umwandlung setzte stets im Innern der Dendriten ein und wurde von den Kanten verlangsamt. Man könnte meinen, dass die Bi-reichen äusseren Grenzen als „Isolatoren" wirksam wären; da aber die

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[1]) Das auf elektrolytischem Wege nach MASON und FORGENG hergestellte Zinn enthielt, wie sie mitteilten, noch geringe Spuren Kupfer, Blei und Eisen. Indes ergab sich, dass der Zusatz viel grösserer Mengen dieser Metalle zum Bi-freien Zinn, als in dem vom Bureau of Standards analysierten Material (vergl. oben § 9) zugegen war, weder nach dem Abschrecken noch nach dem Tempern zur Bildung irgend welcher Strukturen Anass gab. So hatte z.B. ein Zusatz von 0.001 bezw. von 0.01, 0.1, 1.0 Gew. Proz. Blei keinen Einfluss. Wir beabsichtigen in einer späteren Abhandlung auf diese Tatsache zurückzukommen.

Kristallisation schnell vor sich geht und einen ausgeprägten Temperaturgradienten aufweist, wachsen die primären Kristalle fortwährend, so dass das Auftreten von isolierten Körnern von vornherein unwahrscheinlich ist. In dieser Weise kann die Umwandlung durch die ganze Masse fortschreiten.

12. Nach dem von MASON und FORGENG beim Bi Beobachteten liegt es auf der Hand die anderen von uns studierten Metalle, welche in Spuren eine Verzögerung des Fortschreitens der Zinnpest herbeizuführen im stande sind, von dem neu gewonnenen Standpunkt eingehend zu studieren, da derselbe nicht allein für die Kenntnis der U.G. polymorpher Metalle, sondern ebenfalls für die der Korrosionserscheinungen im Allgemeinen von grösster Bedeutung sein dürfte.

Wir hoffen darüber demnächst zu berichten.

ZUSAMMENFASSUNG.

Die vorliegende Untersuchung ergab, dass auch der Zusatz von äusserst geringen Mengen Magnesium zum weissen Zinn die akute Zinnpest hervorzurufen im stande ist; auch hier lässt sich das Eintreten des HEYN–WETZELeffekts als Erklärung heranziehen.

Die enorm verzögernde Wirkung von äusserst geringen Spuren Wismut auf das Eintreten der Zinnpest findet ihre Erklärung in den von MASON und FORGENG beobachteten Erscheinungen, welche die Kristallisation der Zinn-Wismutlegierung begleiten.

Utrecht, August 1938. VAN 'T HOFF-*Laboratorium.*

Mathematics. — *Rectilinear congruences in the three-dimensional projective space built up of quadratic reguli.* By W. VAN DER WOUDE and J. J. DRONKERS.

(Communicated at the meeting of September 24, 1938.)

Of the above-mentioned congruences we wish to prove a few properties and to mention special cases which to us seem to be interesting.

§ 1. We consider a congruence $\mathfrak{K}$, built up of quadratic reguli λ. Simultaneously with this one a second congruence $\mathfrak{K}^+$ is produced, constructed of quadratic reguli λ^+, both λ and λ^+ always lying on the same quadratic surface L; λ and λ^+ are called complementary reguli, $\mathfrak{K}$ and $\mathfrak{K}^+$ complementary congruences.

Theorem. *On the surface* L, *the enveloping surface of system* L, *the focal curves of the two congruences* $\mathfrak{K}$ *and* $\mathfrak{K}^+$ *are lying.*

Proof. L is produced as the locus of the curve (C_4), the characteristic of a surface L. Let us assume for the present that (C_4) has not degenerated. It is then intersected in two points by each straight line lying on L, whether the latter belongs to λ or λ^+. In each point of (C_4) L is touched by L; if b is a straight line on L, then b touches L in the above-mentioned points of intersection and consequently b is a double tangent to L. The two focal curves of $\mathfrak{K}$ and likewise those of $\mathfrak{K}^+$, therefore, lie on L.

§ 2. Before proceeding we shall establish the results analytically. The two congruences are at the same time represented by

$$\varrho\, x_i = a_i(w) + b_i(w)\, v + c_i(w)\, u + d_i(w)\, uv \qquad (i = 1, \ldots, 4),$$

or still more briefly by

$$\varrho\, x = a(w) + b(w)\, v + c(w)\, u + d(w)\, uv \quad . \quad . \quad . \quad . \quad (1)$$

The quadratic surfaces L are indicated by $w = w_0$ (constant), the straight lines of congruence $\mathfrak{K}$ by $w = w_0$, $v = v_0$, those of $\mathfrak{K}^+$ by $w = w_0$, $u = u_0$. The straight lines of $\mathfrak{K}$ are called u-lines, those of $\mathfrak{K}^+$ v-lines.

Now let us determine the focal curves of $\mathfrak{K}$. If in (1) we consider

w and v as functions of u, then a point is determined on each u-line; we need merely express that

$$x^{(0)} (= a(w) + b(w) v),\ x \text{ and } \frac{dx}{du}\left(= \frac{\partial x}{\partial u} + \frac{\partial x}{\partial v} \cdot \frac{dv}{du} + \frac{\partial x}{\partial w} \frac{dw}{du}\right)$$

lie in a straight line. Consequently:

$$\lambda x^{(0)} + \mu x + \nu \left(\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v} \cdot \frac{dv}{du} + \frac{\partial x}{\partial w} \cdot \frac{dw}{du}\right) = 0;$$

The three terms $\lambda x^{(0)}$, μx and $\nu \dfrac{\partial x}{\partial u}$ may be replaced by $\varrho x + \sigma \dfrac{\partial x}{\partial u}$. Then we find from these four equations:

$$\left| x, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial x}{\partial w} \right| = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation both in u and v is of the second (only apparently of the third) degree. It shows that the focal curves of $\mathfrak{R}$ and $\mathfrak{R}^+$ lie on the same surface, determined by (1) and (2).

It is equally simple to prove that (1) and (2) determine the enveloping surface $\mathbf{L}$ of the system L, or — if w is regarded as a constant (w_0) — represent the characteristic of the surface $w = w_0$.

§ 3. In the above it has been assumed that (C_4) had not degenerated. However, that is not essential in the proof of the theorem in § 1. In any case, i. e. also if (C_4) has degenerated, each point of (C_4) is a tangent point of L and $\mathbf{L}$ and consequently the two straight lines of L through that point touch $\mathbf{L}$. This supposition, however, made it clear that the locus of the focal curves of $\mathfrak{R}$ and that of the focal curves of $\mathfrak{R}^+$ is the same. Expressed in the usual terminology of differential geometry: the two congruences have the same focal surfaces (the two branches of $\mathbf{L}$ being regarded as different surfaces).

Now it is remarkable that it would be incorrect to say that the two complementary congruences *always* have the same focal surfaces. That is, at any rate in a literal sense, not always the case.

Let us suppose that each characteristic (C_4) has degenerated into a (C_3) and a straight line b which belongs to $\mathfrak{R}$. Each straight line of a regulus λ will then in two points intersect the curve (C_3) which lies on the same surface L; each straight line of λ^+ intersects (C_3) once and has a point in common with b.

The locus of (C_3) is a surface S, that of b a ruled surface R; together they form $\mathbf{L}$.

Consequently R and S are the focal surfaces of $\mathfrak{R}^+$, both focal curves of $\mathfrak{R}$ lie on S; the ruled surface R belongs to $\mathfrak{R}$.

We may, for example, obtain this case by starting from a ruled surface R:

$$\varrho\, x^{(0)} = a(w) + c(w)\, u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is obvious that the u-curves are the generators of the surface, while of the w-curves three may be taken arbitrarily and the others are determined by the necessity to intersect the generators in projective ranges of points. Congruence $\mathfrak{R}^+$ is produced by the tangents to the w-curves and both congruences are represented by:

$$\varrho\, x = a(w) + c(w)\, u + a'(w)\, v + c'(w)\, uv. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As before, the v-lines form congruence $\mathfrak{R}^+$, the u-lines congruence $\mathfrak{R}$. The enveloping surface L of the quadratic surfaces $L(w = w_0)$ is found from (4) and

$$v \,|\, a + cu, \quad c + c'\, v, \quad a' + c'\, u, \quad a'' + c''\, u \,| = 0$$

i. e. L has fallen apart into the ruled surface R and another surface S, defined by:

$$|\, a, \quad c, \quad a' + c'\, u, \quad a'' + c''\, u \,| + |\, a + cu, \quad c', \quad a', \quad a'' + c''\, u \,|\, v = 0. \quad (5)$$

If, however, in (4) and (5) we consider w as a constant, then (4) and (5) represent a curve of the third degree,

$$\varrho\, x = a(w) + c(w)\, u + a'(w)\, v + c'(w)\, uv$$

as becomes apparent by resolving v from (5) and substituting it in (4); with the straight line $v = 0$ this (C_3) forms the characteristic of the surface L under discussion.

§ 4. We may go yet a step further and cause the enveloping surface of the quadratic surfaces $L(w = w_0)$ to fall apart into surfaces, each of which is a focal surface of $\mathfrak{R}$ or $\mathfrak{R}^+$, none of them being a focal surface of both at the same time. For this purpose we make the characteristic of L degenerate always into 4 straight lines; it is a well known fact that, if two straight lines of λ have part in this characteristic, then two of λ^+ belong to it as well. Each straight line of λ^+ intersects the first pair, each straight line of λ the second. The locus of the first pair consequently consists of the focal surfaces of $\mathfrak{R}^+$, the second pair produces those of $\mathfrak{R}$.

Consequently here belong the focal surfaces of $\mathfrak{R}$ *to* $\mathfrak{R}^+$ *and vice versa.*

Again it is easy to give a (well known) example. Let us start once more from

$$\varrho\, x^{(0)} = a(w) + c(w)\, u \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Here it was required of the w-curves that they intersect the generators in projective ranges. It is well known that the curved asymptotic lines of an arbitrary ruled surface possess this property. Let then the w-curves be curved asymptotic lines of a ruled surface R. Now we consider again the two congruences from

$$\varrho\, x = a(w) + c(w)\, u + a'(w)\, v + c'(w)\, uv \quad . \quad . \quad . \quad . \quad (7)$$

The v-lines forming the congruence $\mathfrak{R}^+$ are now the asymptotic tangents of R. It is known that then the two focal surfaces of $\mathfrak{R}^+$ coincide in R. A double counting straight line of R, therefore, is part of the characteristic of each quadratic surface $L(w = w_0)$ from (4); the remainder of the characteristic consequently must consist of two asymptotic tangents. That is indeed the case. It is namely well known (see e. g. E. P. LANE, Differential Geometry of Curves and Surfaces, Ch. II, Ruled Surfaces) that such a surface $L(w = w_0)$, in other words a quadratic ruled surface described by the asymptotic tangents in the points of a fixed degenerator b, osculates R; i. e. each of these tangents has a contact of the second order with R (usually expressed: has three consecutive points in common with R). However, there are two points on b where the asymptotic tangent has four points in common with R; those points are called the *flecnodal points* and these tangents the *flecnodal tangents*. These two flecnodal tangents together with the double counting straight line b form the characteristic of L. Each of these flecnodal tangents has as locus a ruled surface. These two ruled surfaces consequently are the focal surfaces of $\mathfrak{R}$; they are called the *flecnodal transforms* of R.

It is quite easy to confirm these results analytically. It should merely be borne in mind that the w-curves are asymptotic curves on R, so that in (5):

$$N = |\, a, \quad c, \quad a' + c'\, u, \quad a'' + c''\, u \,| = 0,$$

and proved that then the two values of u from

$$|\, a + c\, u, \quad c', \quad a', \quad a'' + c''\, u \,| = 0$$

indicate the flecnodal points.

§ 5. A second special case — the curve (C_4) need not have degenerated this time — is found by taking linear functions of w in (1) for $a, \ldots d$. Then by

$$\varrho\, x = A + Bu + Cv + Dw + Evw + Fwu + Guv + Huvw, \quad . \quad (8)$$

$A \ldots H$ being constants, three rectilinear congruences are given, of u, v and w-lines respectively, which two by two may be considered as complementary. The three systems of quadratic ruled surfaces

$$u = u_0; \quad v = v_0; \quad w = w_0$$

then have the same enveloping surface L and on L the focal curves of all three congruences are lying.

A simple example — the introduced metrical elements may easily be replaced by projective ones — is the following.

We consider the system of normals of the hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

or, in homogeneous coordinates, of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-t^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

A normal is defined by the points (x_0, y_0, z_0, t_0), whose coordinates satisfy (9), and $\left(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2}, 0\right)$.

If, therefore, we represent (9) by

$$x=\frac{1}{a}(vw+1),\ y=\frac{1}{b}(v-w),\ z=\frac{1}{c}(vw-1),\ t=v+w, \quad . \quad (10)$$

then the congruence of the normals is determined by:

$$\left.\begin{aligned} x &= \left(a+\frac{u}{a}\right)(vw+1)\\ y &= \left(b+\frac{u}{b}\right)(v-w)\\ z &= \left(c+\frac{u}{c}\right)(vw-1)\\ t &= (v+w) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

i. e. the congruence of the normals is formed by the u-lines from (11). At the same time, however, (11) represents two other congruences, viz. the following.

The normals, erected in the points of a straight line m of (9) form a quadratic regulus λ, lying on a hyperbolic paraboloid. On that surface a second regulus λ^+ lies, of which e. g. the above-mentioned straight line m of (9) forms part. These reguli λ^+, m taking the place of every straight line of (9), build up the two congruences of v- and w-lines from (11).

These congruences may be characterized in another way. An arbitrary straight line is projected on a surface of the second degree in a (C_4). However, every v- or w-line from (11) is intersected by the normals in the points of a straight line of (9).

The v- and w-lines from (11) consequently are projected on (9) in a straight line and a (C_3).

In connection with § 1 we find from this:

The straight lines, whose projections on a surface of the second degree O_2 *(of rank four) have degenerated into a straight line and a* (C_3), *form two congruences; these lines, like the normals of* O_2, *are double tangents to the surface of the centres of principal curvature of* O_2.

It is easy to be seen that these two congruences have the same degree and class as the congruence of normals, in which case these numbers are 6 and 2. For example, the number of value systems of u, v, w which satisfies (11), the ratios of x, y, z, t being considered as given, indicates the degree of all three congruences.

The two congruences mentioned above are of the sixth degree and the second class.

Botany. — *On the relation between internal and external medium in Artemia salina (L.) var. principalis* SIMON. By HERBERT WARREN, DONALD KUENEN and L. G. M. BAAS BECKING. (From the Botanical Institute, University of Leyden.)

(Communicated at the meeting of September 24, 1938.)

The unique osmotic regulation of Artemia has been commented upon several times, but accurate data on this osmoregulation are still lacking. It seemed, therefore, worth while to investigate the relations between internal- and external milieu in this curious phyllopod. Two of us started to work on this problem at Pacific Grove, California, already in 1929, while the experiments are being continued now at Leyden.

In the earlier experiments living material could be obtained from a near-by salt work, while at Leyden we raise the animals from eggs, collected by one of us in California in 1930. More than 10 % of these eggs are still viable.

The problem was tackled by means of different methods:

1. Direct analysis of the haemocele fluid.
2. Determination of the refractive index of the haemocele fluid.
3. Determination of water endosmosis or exosmosis on transfer from one salt concentration to the other.
4. Volumenometry.

1. *Direct analysis of the haemocele fluid.*

One large female may yield as much as 5.9 mm^3 of haemocele fluid, which may be obtained by means of a fine capillary tube. On the average, adult specimens may give 5—6 mm^3. We worked chiefly with animals grown in a brine of a s.g. 1.075/15° ($n_D^{25} = 1.3518$). The following table gives the data obtained.

Blood of:	1 ♂	15 ♂	3 ♂	8 ♀	6 ♀	10 ♀	Weighted averages
specific gravity:	—	—	—	1.032	1.044	—	1.037
% dry weight:	4.85	4.83	6.33	—	5.15	6.34	5.36
% Na:	.92	.75	—	—	—	.84	.81[5]

The average weight of a male proved to be 7.5 mg and that of a female 9.2 mg. The average dry weight amounted to 7.48 % of the total body weight. If the Na present in the blood were calculated as NaCl, we would obtain 2.65 % NaCl, while the external milieu contained about 9 % NaCl.

From 9 females grown in a brine of s.g. 1.050 the Na- and chloride contents were determined separately. We found Na .67 % and Cl .87 %.

This shows that other chlorides except NaCl, or other Na-compounds except chlorides, should be present as the Cl corresponds to only .34 % Na. The variability, however, seems extremely large as the blood of 15 males, grown in a brine of s.g. 1.050 yielded only .68 % ash, of which ash 70 % consisted of NaCl.

MEDWEDEWA, using BARGER's method, found the haemocele of Artemia, grown in a brine of 8° B (s.g. ± 1.060) isotonic with 1.3 % NaCl, while blood from animals from a brine of 4.5 ° B (s.g. ± 1.033) proved to be isotonic with 1.2 % NaCl.

If we assume the excess sodium to be osmotically less active we obtain (taking the proportion Na : Cl = 67 : 87)

at s.g. 1.075 NaCl 1.75 % of haemocele
at s.g. 1.050 „ 1.21 % „ „

This corresponds well with the values found by MEDWEDEWA.

As compared with other crustaceans Artemia seems to have a similar NaCl percentage in the ash (Artemia 68, Potamobius 50, Limulus 83, see VON FÜRTH), while the NaCl contents seems low (Artemia 1.21—1.75 %, Ecrevissia 2.94, Carcinus 2.70, Potamobius 1.73, Nephrops 2.77, see FRÉDERICQ).

2.

By means of a fine capillary pipette blood may be obtained from carefully cleaned and dry animals from the dorsal side. The refractive index of the blood of a single animal is easily determined by means of an Abbe refractometer at 25° C.

The refractive index of the blood proved to be variable, but markedly influenced by the salinity of the external milieu.

In statistical treatment differences in the fourth decimal place of the refractive index were used as class-differences. The following table summarizes the results.

N_{25}^{D} external milieu	N_{25}^{D} Mean internal milieu (haemocele fluid)	Number of variants	σ Mean in classes (percentual distribution)
1.3325	1.3366	16	0.93
1.3389	1.3377	31	1.15
1.3470	1.3382	35	1.15
1.3525	1.3387	86	1.00
1.3563	1.3416	20	1.35
1.3620	1.3407	43	2.74
1.3770	1.3518	25	8.55

The results are, moreover, represented on the accompanying graph. Here the abscissa gives the N_D^{25} of the haemocele fluid, the ordinate shows this constant for the external milieu. The line marked "no exchange" indicates equality between external and internal milieu.

The heavy line gives the smoothed curve of the averages. Extreme variation is indicated by the broken boundaries. The circles between A and B indicate values obtained by ourselves and by MEDWEDEWA for NaCl, showing that while, as found in the previous paragraph, the NaCl seems

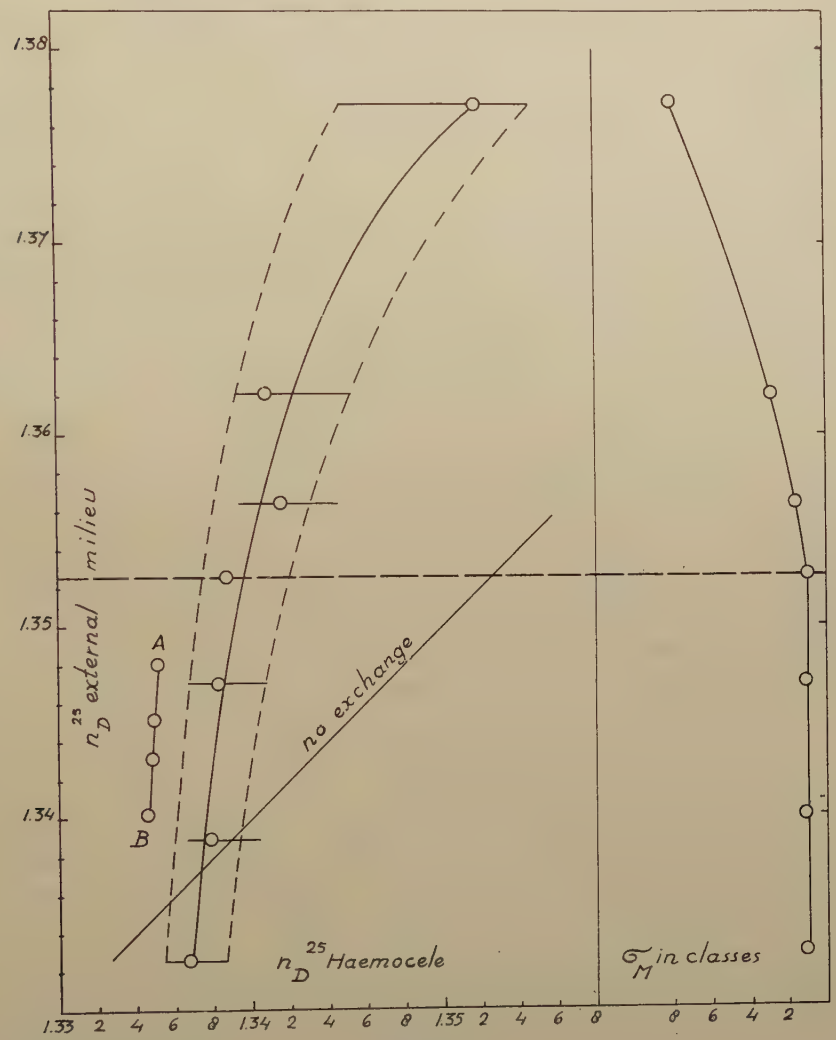

Fig. 1.

to account almost quantitatively for the osmotic pressure, the average refractive index of the haemocele fluid appears to be higher.

The line of the average refractive index seems, at first, very little influenced by the salinity of the environment. Later, above about 2.0 molar

(external environment) the influence appears to be much more marked. The limit is indicated in the graph by a horizontal, broken line.

In the above table the variability of the refractive index values seems to remain almost constant to the same value in the outer environment, while at higher salinities it increases. This is depicted on the right-hand side of the graph. Both results seem to indicate that active regulation is most efficient up to about 2.0 mol. NaCl in the outer fluid, after which the blood seems increasingly influenced by an increase in salinity in the outer environment.

It should be stated that, assuming the index of refraction to be a linear function of the concentration, the concentration of the "non-osmotic components" remains constant in an outer environment of 0—2.0 mol NaCl.

3.

From the graph it appears that, if the percentage of solid matter in the body fluid be proportional to its refractive index minus 1.3325 transfer from low to high concentration will cause a considerable loss of fluid from the body. At about 2 molar NaCl in the external environment we know the blood to contain about 5 % of dry matter.

As the NaCl contents at 1 molar NaCl in the external milieu is only 80 % of that at 2 molar NaCl we may assume the dry matter in the blood at 1 molar NaCl in the external milieu to amount to 4 %.

From the assumed linear proportionality of (N^{D}_{25} — 1.3325) and solid matter in the blood, we might conclude that the body fluid of animals kept in 5 molar NaCl should contain about $4 \times 4 = 16$ % solid matter.

As one animal might contain about 6 mm^3 of available haemocele fluid the animals in 1 molar NaCl contain, $96 \times 6 = 5.76$ mm^3 water. If the animals be transferred to 5 molar NaCl, where they should contain ± 16 % solid matter and if the amount of water left in the body be called x mm^3 the relation $\frac{4 \times 6 \,.\, x}{x} = 16$ should hold. Accordingly $x = 1.60$. Therefore 5.76 — 1.6 or ± 4 mm^3 of water should be excreted.

If one animal be transferred from 1 molar NaCl into 1 cc of 5 molar NaCl the molarity should be lowered by 0.04 corresponding to a lowering of 2.64 units in the fourth decimal place of the refractometer. It seemed feasible, therefore, to study exchange in this way.

For practical reasons not less than one cc of liquid should be used as the animals, even in aerated brine, do not seem to survive.

As salt, given off by the animals might obscure the result, Artemiae were transferred from brine into distilled water. On the average, (27 animals studies in 6 series) one animal increased the refractive index of 1 cc of water by 0.25 in the fourth decimal place.

Transfer of 36 animals in 11 series from 1 molar to 5 molar brine yielded an average decrease per animal per cc of 1.15 units in the fourth decimal place of N^{25}_{D}. To this value should be added the correction 0.25,

as this correction runs in the opposite direction from our base-value. Therefore the decrease of N_D^{25}/animal/cc amounted to 1.4 units in the fourth recimal place. As 0.91 units correspond to a decrease in molarity of 0.01, 1.4 units signify a decrease in molarity of 1.54, corresponding to an secretion of x mm^3 of water.

The following relation should hold:

$$1.00\,x = \frac{5.0000}{4.9846} \text{ or } x = 3.1.$$

As a more sensitive dip-refractometer could not be used there seems to be a fair agreement between the value of 4 mm^3 calculated and 3.1 mm^3 found in the experiment. If other substances besides pure water were excreted the discrepancy would be even less.

4.

As osmotic swelling and shrinkage of the animals was repeatedly observed in transfers of the animals from brines of higher to those of lower density and vice versa, attempts were made to determine the shrinkage and the swelling of the animals. To this end a simple dilatometer was designed by Mr. E. HANSON, which we intend to describe elsewhere.

The following results were obtained.

Numbers of animals	Transfer from	To	Average % swelling	Calculated %	Average % shrinkage
9	1 molar	3 molar	—	(80)	19
8	1 „	2 „	—	(24)	12
8	2 „	1 „	11	(19)	—
5	2½ „	½ „	11	(42)	—
5	1½ „	½ „	7	(20)	—
5	½ „	1½ „	—	(28)	3
2	3 „	1 „	10	(41)	—
5	2 „	0 „	10	(37)	—
4	2½ „	1½ „	7	(28)	—

The numbers in the column "% swelling calculated" were obtained in the following way.

If, after uptake or excretion of water, the percentage in dry weight changes from a_1 to a_2, the water-content will change from $(100 - a_1)$ to $(100 - a_2)$ %. The procentual swelling or shrinkage will amount to,

$$(100 - a_1)\frac{a_2}{a_1} - 100 \quad \text{resp.} \quad 100\left(\frac{a_2}{a_1} - 1\right) - a_2.$$

From the table it follows that the observed swelling and shrinkage is much less than predicted by the above calculation (up to ¼th of the calculated value). It is very well possible that part of the water excreted by the body fluid remains in the gut, or inversely, the animal takes up water out of the gut, which would make no difference in the total volume. As the intestinal tract occupies a large percentage of the body-volume, the above assumption might well account for this discrepancy.

CONCLUSIONS.

1. The osmoregulation of Artemia salina seems very efficient up to concentrations isotonic with 2.0 mol. NaCl.
2. The NaCl-content of the blood may account quantitatively for its osmotic properties.
3. There are indications that regulation is effected by means of excretion or uptake of water.
4. Swelling and shrinkage of the animals as a response to osmotic gradient is only about ¼th of the predicted values. Exchange between haemocoele and gut may account for this fact.

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Beweis von Satz 4. Wegen (7) hat man

$$L_{k,m}(z\,e^{-\pi i}) = e^{-(2m+1)\pi i}\,L_{k,m}(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Sind die Bedingungen (14), (15), (19) und (20) mit $-k$ statt k erfüllt, so folgt also aus (18), mit $-k$ statt k, $z=\zeta\,e^{\frac{1}{2}\pi i}$ $(-\frac{1}{4}\pi < arg\,\zeta < \frac{1}{4}\pi)$, $\tau=\frac{1}{2}\pi - arg\,z = -arg\,\zeta$ und $u=v\,e^{-\pi i}$ $(arg\,v=\pi+arg\,u=\pi+\tau=\pi-arg\,\zeta)$ angewendet,

$$\left.\begin{aligned} e^{k\pi i}\,T_{-k,m}(\zeta\,e^{\frac{1}{2}\pi i}) = {} & \frac{4i\,e^{-\frac{1}{2}(m+3k+2\alpha+\frac{1}{2})\pi i}\,\zeta^{m+k+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k+m)} \\ & \times\int_{\infty\,e^{i(\pi-arg\,\zeta)}}^{0} L_{-m-\alpha,k+\alpha}(v)\,K_{m-k-2\alpha-\frac{1}{2}}(2\zeta\,v\,e^{-\frac{1}{2}\pi i})\,v^{m+k-\frac{1}{2}}\,dv \end{aligned}\right\} . \quad (96)$$

Aus (18), mit $-k$ statt k, $z=\zeta\,e^{-\frac{1}{2}\pi i}$ $(-\frac{1}{4}\pi < arg\,\zeta < \frac{1}{4}\pi)$, $\tau=-\frac{1}{2}\pi - arg\,z = -arg\,\zeta$ und $u=v$ angewendet, ergibt sich auch

$$\left.\begin{aligned} -e^{-k\pi i}\,T_{-k,m}(\zeta\,e^{-\frac{1}{2}\pi i}) = {} & \frac{4i\,e^{-\frac{1}{2}(m+3k+2\alpha+\frac{1}{2})\pi i}\,\zeta^{m+k+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k+m)} \\ & \times\int_{0}^{\infty\,e^{-i\,arg\,\zeta}} L_{-m-\alpha,k+\alpha}(v)\,K_{m-k-2\alpha-\frac{1}{2}}(2\zeta\,v\,e^{-\frac{1}{2}\pi i})\,v^{m+k-\frac{1}{2}}\,dv \end{aligned}\right\} . \quad (97)$$

Durch Addition von (96) und (97) erhält man mit Rücksicht auf (70)[52])

$$\left.\begin{aligned} T_{k,m}(\zeta) = {} & \frac{2\,e^{-\frac{1}{2}(m+3k+2\alpha+\frac{1}{2})\pi i}\,\zeta^{m+k+2\alpha+3/2}\,e^{\zeta/2}\,\Gamma(\frac{1}{2}+k-m)}{\pi} \\ & \times\int_{\infty\,e^{i(\pi-arg\,\zeta)}}^{\infty\,e^{-i\,arg\,\zeta}} L_{-m-\alpha,k+\alpha}(v)\,K_{m-k-2\alpha-\frac{1}{2}}(2\zeta\,v\,e^{-\frac{1}{2}\pi i})\,v^{m+k-\frac{1}{2}}\,dv \end{aligned}\right\} . \quad (98)$$

(bei dieser Integration wird der Punkt $v=0$ vermieden durch einen oberhalb dieses Punktes liegenden Halbkreis).

Nun folgt aus (79), (73) und (74)[53]), dass das auf der rechten Seite von

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[51]) Erste und zweite Mitteilung: Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 624—633 und 744—755 (1938).

[52]) Ich nehme an, dass $m-k \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ ist.

[53]) Das Verhalten von $L_{k,m}(z)$ für grosse Werte von $|z|$ mit $\frac{3}{4}\pi < arg\,z < \frac{5}{4}\pi$ folgt mittels (95) aus dem Verhalten dieser Funktion für $-\frac{1}{4}\pi < arg\,z < \frac{1}{4}\pi$.

(98) vorkommende Integral (mit $-\frac{1}{4}\pi < arg\,\zeta < \frac{1}{4}\pi$) für $\Re(1-k-3m-2\alpha) > 0$ eine analytische Funktion von k, m und α ist [54]). Da auch die linke Seite eine analytische Funktion von k, m und α ist, so gilt Formel (98) nach der Theorie der analytischen Fortsetzung für alle Werte von k, m und α mit $m-k \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ und $\Re(1-k-3m-2\alpha) > 0$. Hiermit ist der Beweis von Satz 4 geliefert, denn wegen (2) gilt

$$K_{m-k-2\alpha-\frac{1}{2}}(2\zeta v e^{-\frac{1}{2}\pi i}) = \tfrac{1}{2}\pi e^{\frac{1}{2}(m-k-2\alpha+\frac{1}{2})\pi i} H^{(1)}_{m-k-2\alpha-\frac{1}{2}}(2\zeta v).$$

Der Beweis der Bemerkung zu Satz 4 ist demjenigen der Bemerkung zu Satz 3 analog.

Beweis von Satz 5. Ich nehme an, dass k, m und α den Bedingungen (37), (40) und (41) genügen und dass überdies $\Re(\frac{1}{2}+k-m) > 0$ ist; die Beziehungen (14), (15), (19) und (20) sind also mit $-k$ statt k und $-m$ statt m erfüllt. Ist ausserdem $\frac{1}{4}\pi < arg\,z < \frac{3}{4}\pi$, bezw. $-\frac{3}{4}\pi < arg\,z < -\frac{1}{4}\pi$, so darf man in (18) (mit $-k$ statt k und $-m$ statt m angewendet) $\tau = \frac{1}{2}\pi - arg\,z$, bezw. $\tau = -\frac{1}{2}\pi - arg\,z$ setzen (siehe (16) und (17)). Ersetzt man ferner noch z durch $ze^{\frac{1}{2}\pi i}$, bezw. durch $ze^{-\frac{1}{2}\pi i}$, so findet man [55])

$$\left.\begin{aligned} T_{-k,-m}(z e^{\frac{1}{2}\pi i}) = {} & \frac{4 e^{\frac{1}{2}(k-m+2\alpha+3/2)\pi i}\, z^{k-m+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k-m)} \\ & \times \int_0^{\infty e^{-i\arg z}} L_{m-\alpha,k+\alpha}(u)\, K_{m+k+2\alpha+\frac{1}{2}}(2zue^{\frac{1}{2}\pi i})\, u^{k-m-\frac{1}{2}}\, du, \end{aligned}\right\} \quad (99)$$

bezw.

$$\left.\begin{aligned} T_{-k,-m}(z e^{-\frac{1}{2}\pi i}) = {} & \frac{4 e^{-\frac{1}{2}(k-m+2\alpha+3/2)\pi i}\, z^{k-m+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k-m)} \\ & \times \int_0^{\infty e^{-i\arg z}} L_{m-\alpha,k+\alpha}(u)\, K_{m+k+2\alpha+\frac{1}{2}}(2zue^{-\frac{1}{2}\pi i})\, u^{k-m-\frac{1}{2}}\, du, \end{aligned}\right\} \quad (100)$$

und in diesen beiden Relationen ist $|arg\,z| < \frac{1}{4}\pi$.

Nun ist [56])

$$K_\nu(we^{\frac{1}{2}\pi i}) = -\tfrac{1}{2}\pi i e^{-\frac{1}{2}\nu\pi i} H^{(2)}_\nu(w) \quad (101)$$

und

$$K_\nu(we^{-\frac{1}{2}\pi i}) = \tfrac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i} H^{(1)}_\nu(w). \quad (102)$$

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[54]) Ich brauche nur die Konvergenz für $|v| = \infty$ zu untersuchen; denn der Punkt $v = 0$ liegt nicht auf dem Integrationswege.

[55]) Ich benutze die bekannte Relation $K_\nu = K_{-\nu}$.

[56]) Formel (102) ist gleichbedeutend mit (2); Beziehung (101) folgt aus (102) und (52).

Wegen $T_{-k,m} = T_{-k,-m}$ folgt also aus (99)

$$e^{m\pi i}\, T_{-k,m}(z e^{\frac{1}{2}\pi i}) = \frac{2\pi z^{k-m+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k-m)} \int_0^{\infty e^{-i\arg z}} L_{m-\alpha,k+\alpha}(u)\, H^{(2)}_{m+k+2\alpha+\frac{1}{2}}(2zu)\, u^{k-m-\frac{1}{2}}\, du\,; \quad (103)$$

ebenso aus (100)

$$e^{-m\pi i}\, T_{-k,m}(z e^{-\frac{1}{2}\pi i}) = \frac{2\pi z^{k-m+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k-m)} \int_0^{\infty e^{-i\arg z}} L_{m-\alpha,k+\alpha}(u)\, H^{(1)}_{m+k+2\alpha+\frac{1}{2}}(2zu)\, u^{k-m-\frac{1}{2}}\, du. \quad (104)$$

Integraldarstellung (38) ergibt sich jetzt aus (71), (103) und (104), so dass (38) bewiesen ist für den Fall, dass $\Re(\frac{1}{2}+k-m) > 0$ ist. Durch analytische Fortsetzung schliesst man aber (siehe (75), (78), (73), (74) und (80)), dass (38) gilt für alle Werte von k, m und α mit (40) und (41) (Bedingung (37) folgt durch Addition aus (40) und (41)). Hiermit ist Satz 5 bewiesen.

Der Beweis der Bemerkung zu Satz 5 ist demjenigen der Bemerkung zu Satz 3 analog.

Beweis von Satz 6 [57]). Sind die Voraussetzungen (21), (23), (24) und (25) mit $-k$ statt k erfüllt, so folgt aus (22) (mit $-k$ statt k, $z = \zeta e^{\frac{1}{2}\pi i}$ $(\zeta > 0)$ und $u = v e^{-\frac{1}{2}\pi i}$ angewendet) wegen $T_{m+\alpha,-k-\alpha} = T_{m+\alpha,k+\alpha}$

$$e^{m\pi i}\, T_{-k,m}(\zeta e^{\frac{1}{2}\pi i}) = \frac{2i e^{(m+\alpha)\pi i}\, \zeta^{m+k+2\alpha+3/2}\, \Gamma(\frac{1}{2}-k-m-2\alpha)}{\Gamma(\frac{1}{2}+k+m)} \times \int_0^{\infty} T_{m+\alpha,k+\alpha}(v e^{-\frac{1}{2}\pi i})\, J_{m-k-2\alpha-\frac{1}{2}}(2\zeta v)\, v^{m+k-\frac{1}{2}}\, dv.$$

Ich benutze nicht nur diese Formel, sondern auch die Beziehung, die ich erhalte, wenn ich i darin durch $-i$ ersetze. Diese zwei Formeln liefern bei Addition (man beachte (71)) [58])

$$L_{k,m}(\zeta) = \frac{\zeta^{m+k+2\alpha+3/2}\, \Gamma(\frac{1}{2}+k-m)\, \Gamma(\frac{1}{2}-k-m-2\alpha)}{\pi i\, \Gamma(\frac{1}{2}+k+m)} \times \int_0^{\infty} \{e^{-(m+\alpha)\pi i}\, T_{m+\alpha,k+\alpha}(v e^{\frac{1}{2}\pi i}) - e^{(m+\alpha)\pi i}\, T_{m+\alpha,k+\alpha}(v e^{-\frac{1}{2}\pi i})\} \times J_{m-k-2\alpha-\frac{1}{2}}(2\zeta v)\, v^{m+k-\frac{1}{2}}\, dv = \frac{2\zeta^{m+k+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k+m)} \times \int_0^{\infty} e^{-v^2}\, T_{-m-\alpha,k+\alpha}(v)\, J_{m-k-2\alpha-\frac{1}{2}}(2\zeta v)\, v^{m+k-\frac{1}{2}}\, dv$$

wegen (70) (mit $-m-\alpha$ statt k, $k+\alpha$ statt m und v statt z angewendet).

[57]) Man kann Satz 6 auch mit Hilfe der HANKELschen Transformation aus Satz 3 ableiten.

[58]) Ich nehme an, dass $m-k \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ ist.

Sind die Bedingungen (21), (23), (24) und (25) mit $-k$ statt k erfüllt und ist $m-k \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, z > 0$ und $\tau = 0$, so ist also der Beweis von (42) geliefert. Aus (76), (78), (72) und (80) [59]) ergibt sich aber, dass die rechte Seite von (42) für beliebiges $z \neq 0$ und alle Werte von k, m, α und τ mit $\Re(\frac{1}{2}+k+m) > 0$, $\Re(\frac{1}{2}-k+m-2\alpha) > 0$ und $-\frac{1}{4}\pi < \tau < \frac{1}{4}\pi$ existiert und für diese Werte von τ nicht von τ abhängig ist. Um Satz 6 im vollen Umfang zu beweisen, braucht man also noch die Theorie der analytischen Fortsetzung anzuwenden.

Beweis von Satz 7. Man hat [60])

$$M_{k,m}(z) = e^{-(m+\frac{1}{2})\pi i} M_{-k,m}(z e^{\pi i}),$$

somit wegen (6)

$$L_{k,m}(z) = e^{-(m+\frac{1}{2})\pi i} e^{-z^2} L_{-k,m}(z e^{\frac{1}{2}\pi i}). \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Formel (43) folgt nun aus (42) (mit $-k$ statt k und $z e^{\frac{1}{2}\pi i}$ statt z angewendet), (1) und (105).

§ 6. Beweis der Sätze 8—11.

Beweis von Satz 8. Ist $\Re(\frac{1}{2}-k\pm m) > 0$, so ist die rechte Seite von (45) wegen (77), (78), (79) und (80) eine analytische Funktion von z für $|\arg z| < \frac{1}{2}\pi$ und eine stetige Funktion von z für $|\arg z| \leqq \frac{1}{2}\pi$. Ich darf also $|\arg z| < \frac{1}{4}\pi$ voraussetzen.

Nun hat man [61]), falls $|\arg \eta| < \frac{1}{2}\pi$ und $\Re(\frac{1}{2}-k\pm m) > 0$ ist,

$$\left.\begin{aligned}\int_0^\infty K_{2m}\left(\frac{2v}{\eta}\right) J_{m-k}(v) J_{-m-k}(v)\, dv = \frac{\eta^{1-2k}\,\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}{2^{2-2k}\,\Gamma(1-k+m)\,\Gamma(1-k-m)} \\ \times {}_4F_3\left(\begin{matrix}\frac{1}{2}-k+m, \frac{1}{2}-k-m, \frac{1}{2}-k, 1-k; \\ 1-k+m, 1-k-m, 1-2k; -\eta^2\end{matrix}\right).\end{aligned}\right\} \quad (106)$$

Für $|\arg w| < \frac{1}{4}\pi$ gilt ferner nach Herrn HARDY [62])

$$K_{4m}(4w) = \frac{2}{\pi}\int_0^\infty K_{2m}(2t)\, K_{2m}\left(\frac{2w^2}{t}\right) dt.$$

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59) Siehe auch Fussnote 50).

60) Siehe MEIJER, [26], Formeln (1) und (6).

61) Siehe MEIJER, [27], Hilfssatz 1. Formel (106) ist in [27] nur für $|\eta| < 1$ bewiesen worden; sie gilt aber nach der Theorie der analytischen Fortsetzung für alle Werte von η mit $|\arg \eta| < \frac{1}{2}\pi$.

62) HARDY, [14], 190.

Folglich ist [63])

$$\int_0^\infty J_{m-k}(u^2)\, J_{-m-k}(u^2)\, K_{4m}(4zu)\, u\, du$$

$$= \frac{2}{\pi}\int_0^\infty J_{m-k}(u^2)\, J_{-m-k}(u^2)\, u\, du \int_0^\infty K_{2m}(2t)\, K_{2m}\left(\frac{2z^2u^2}{t}\right) dt$$

$$= \frac{2}{\pi}\int_0^\infty K_{2m}(2t)\, dt \int_0^\infty K_{2m}\left(\frac{2z^2u^2}{t}\right) J_{m-k}(u^2)\, J_{-m-k}(u^2)\, u\, du$$

$$= \frac{2^{2k-2}\, z^{4k-2}\, \Gamma(\frac{1}{2}-k+m)\, \Gamma(\frac{1}{2}-k-m)}{\pi\, \Gamma(1-k+m)\, \Gamma(1-k-m)} \int_0^\infty t^{1-2k}\, K_{2m}(2t)$$

$$\times {}_4F_3\left(\begin{matrix}\frac{1}{2}-k+m,\ \frac{1}{2}-k-m,\ \frac{1}{2}-k,\ 1-k;\\ 1-k+m,\ 1-k-m,\ 1-2k;\end{matrix} -t^2/z^4\right) dt$$

wegen (106) (mit $v = u^2$ und $\eta = \frac{t}{z^2}$ angewendet).

Der letzte Ausdruck ist aber gleich [64])

$$\frac{\Gamma(\frac{1}{2}-k+m)\, \Gamma(\frac{1}{2}-k-m)}{16\pi z^2}\, W_{k,m}(2z^2 e^{\frac{1}{2}\pi i})\, W_{k,m}(2z^2 e^{-\frac{1}{2}\pi i}),$$

so dass der Beweis von Satz 8 geliefert ist.

Beweis von Satz 9. Ist $\zeta > 0$ und $\Re(s) > |\Re(\mu)| + |\Re(\nu)|$, so gilt bekanntlich [65])

$$\int_0^\infty K_\mu(\zeta u^2)\, K_\nu(\zeta u^2)\, u^{2s-1}\, du$$

$$= \frac{\Gamma\left(\frac{s+\mu+\nu}{2}\right) \Gamma\left(\frac{s-\mu+\nu}{2}\right) \Gamma\left(\frac{s+\mu-\nu}{2}\right) \Gamma\left(\frac{s-\mu-\nu}{2}\right)}{2^{4-s}\, \zeta^s\, \Gamma(s)}.$$

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63) Das auf der zweiten Zeile stehende wiederholte Integral ist absolut konvergent, so dass die Vertauschung der Integrationsfolge erlaubt ist.

64) MEIJER, [24], Satz 6.

65) TITCHMARSH, [33], 98; DIXON and FERRAR, [8], 129.

Mittels gliedweiser Integration [66]) findet man also, falls $\zeta > 0$, $\Re(m) > -\frac{1}{4}$ und $\Re(\frac{1}{2} \pm k + m) > 0$ ist,

$$\left.\begin{aligned} &\int_0^\infty K_{m+k}(\zeta u^2)\, K_{m-k}(\zeta u^2)\, J_{4m}(4zu)\, u\, du \\ &= \sum_{h=0}^{\infty} \frac{(-1)^h (2z)^{4m+2h}}{\Gamma(h+1)\,\Gamma(4m+h+1)} \int_0^\infty K_{m+k}(\zeta u^2)\, K_{m-k}(\zeta u^2)\, u^{4m+2h+1}\, du \\ &= \frac{2^{2m-3}\pi z^{4m}}{\zeta^{2m+1}} \sum_{h=0}^{\infty} \frac{(-1)^h (2z^2)^h\, \Gamma(m-k+\frac{1}{2}h+\frac{1}{2})\, \Gamma(m+k+\frac{1}{2}h+\frac{1}{2})}{\zeta^h\, \Gamma(\frac{1}{2}h+1)\, \Gamma(2m+\frac{1}{2}h+1)\, \Gamma(2m+h+1)}. \end{aligned}\right\} \quad (107)$$

Ist $z > 0$, so ist das linkerhand auftretende Integral eine analytische Funktion von ζ für $|\arg\zeta| < \frac{1}{2}\pi$ und eine stetige Funktion von ζ für $|\arg\zeta| \leqq \frac{1}{2}\pi$ (siehe (77), (78), (79) und (80)). Da auch die rechte Seite eine analytische Funktion von ζ ist, so gilt Formel (107) für $|\arg\zeta| \leqq \frac{1}{2}\pi$, wofern $z > 0$ ist.

Nimmt man nun $\zeta = e^{-\frac{1}{2}\pi i}$, bezw. $\zeta = e^{\frac{1}{2}\pi i}$, so erhält man [67])

$$\left.\begin{aligned} &\int_0^\infty H^{(1)}_{m+k}(u^2)\, H^{(1)}_{m-k}(u^2)\, J_{4m}(4zu)\, u\, du \\ &= \frac{(2z^2)^{2m}}{2\pi i} \sum_{h=0}^{\infty} \frac{e^{-\frac{1}{2}h\pi i}\, (2z^2)^h\, \Gamma(m-k+\frac{1}{2}h+\frac{1}{2})\, \Gamma(m+k+\frac{1}{2}h+\frac{1}{2})}{\Gamma(\frac{1}{2}h+1)\, \Gamma(2m+\frac{1}{2}h+1)\, \Gamma(2m+h+1)}, \end{aligned}\right\} \quad (108)$$

bezw.

$$\left.\begin{aligned} &\int_0^\infty H^{(2)}_{m+k}(u^2)\, H^{(2)}_{m-k}(u^2)\, J_{4m}(4zu)\, u\, du \\ &= -\frac{(2z^2)^{2m}}{2\pi i} \sum_{h=0}^{\infty} \frac{e^{\frac{1}{2}h\pi i}\, (2z^2)^h\, \Gamma(m-k+\frac{1}{2}h+\frac{1}{2})\, \Gamma(m+k+\frac{1}{2}h+\frac{1}{2})}{\Gamma(\frac{1}{2}h+1)\, \Gamma(2m+\frac{1}{2}h+1)\, \Gamma(2m+h+1)}. \end{aligned}\right\} \quad (109)$$

Durch Subtraktion von (108) und (109) findet man

$$\left.\begin{aligned} &\int_0^\infty \{H^{(1)}_{m+k}(u^2)\, H^{(1)}_{m-k}(u^2) - H^{(2)}_{m+k}(u^2)\, H^{(2)}_{m-k}(u^2)\}\, J_{4m}(4zu)\, u\, du \\ &= \frac{(2z^2)^{2m}}{\pi i} \sum_{j=0}^{\infty} \frac{(-1)^j (2z^2)^{2j}\, \Gamma(m-k+j+\frac{1}{2})\, \Gamma(m+k+j+\frac{1}{2})}{\Gamma(j+1)\, \Gamma(2m+j+1)\, \Gamma(2m+2j+1)} \\ &= \frac{(2z^2)^{2m}\, \Gamma(\frac{1}{2}-k+m)\, \Gamma(\frac{1}{2}+k+m)}{\pi i\, \Gamma^2(2m+1)}\, {}_2F_3\left(\begin{matrix}\frac{1}{2}-k+m,\ \frac{1}{2}+k+m; \\ 2m+1, m+\frac{1}{2}, m+1;\end{matrix} -z^4\right). \end{aligned}\right\} \quad (110)$$

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66) Gliedweise Integration ist erlaubt; man vergl. BROMWICH, [7], § 176 B.

67) Ich verwende die Beziehungen (102) und (101).

Man hat aber nach Herrn BAILEY [68])

$$ {}_1F_1(a;b;w) \times {}_1F_1(a;b;-w) = {}_2F_3(a, b-a; b, \tfrac{1}{2}b, \tfrac{1}{2}b+\tfrac{1}{2}; \tfrac{1}{4}w^2), $$

somit wegen (3)

$$ M_{k,m}(2z^2 e^{\frac{1}{2}\pi i}) M_{k,m}(2z^2 e^{-\frac{1}{2}\pi i}) = (2z^2)^{2m+1} {}_2F_3\left(\begin{matrix}\frac{1}{2}-k+m, \frac{1}{2}+k+m; \\ 2m+1, m+\frac{1}{2}, m+1; -z^4\end{matrix}\right). \quad (111) $$

Formel (46) folgt nun aus (110) und (111).

Beweis von Satz 10. Ich betrachte für $|\arg z| \leqq \frac{1}{4}\pi$, $\Re(m) > -\frac{1}{4}$ und $\Re(\frac{1}{2}-k+m) > 0$ das Integral

$$ G(z) = \int_0^\infty K_{m+k}\left(\frac{v^2}{z^2}\right) I_{m-k}\left(\frac{v^2}{z^2}\right) J_{4m}(4v)\, v\, dv \quad . \quad . \quad . \quad (112) $$

und werde zeigen, dass

$$ G(z) = \frac{\Gamma(\frac{1}{2}-k+m)}{8\,\Gamma(2m+1)} W_{k,m}(2z^2)\, M_{-k,m}(2z^2) \quad . \quad . \quad . \quad (113) $$

ist.

Nun hat man [69]) für grosse Werte von $|z|$

$$ I_\nu(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \{1 + O(z^{-1})\} + \frac{e^{-z \pm (\nu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}} \{1 + O(z^{-1})\} \ldots \quad (114) $$

Hieraus und aus (77), (78), (79) und (80) geht hervor, dass $G(z)$ eine analytische Funktion von k und m, überdies für $|\arg z| < \frac{1}{4}\pi$ eine analytische Funktion von z und für $|\arg z| \leqq \frac{1}{4}\pi$ eine stetige Funktion von z ist. Ich darf also

$$ z > 0,\ k < \tfrac{1}{4} \text{ und } m > -\tfrac{1}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (115) $$

voraussetzen.

Nun gilt bekanntlich [70]), falls $\zeta > 0$ und $n > -\frac{1}{2}$ ist,

$$ \int_0^\infty J_{2n}(4v) J_n\left(\frac{v^2}{\zeta}\right) v\, dv = \tfrac{1}{2}\zeta J_n(4\zeta) \quad . \quad . \quad . \quad . \quad . \quad (116) $$

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68) BAILEY, [1], 246.

69) WATSON, [37], 203. In (114) gilt das obere oder das untere Zeichen, je nachdem $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ oder $-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi$ ist.

70) HARDY, [14], 190.

Ferner hat man [71]), falls $w > 0$, $\mu + \nu > -1$ und $\mu - \nu < \frac{3}{2}$ ist,

$$K_\mu(w)\, I_\nu(w) = \int_0^\infty J_{\mu+\nu}(2wt)\, \Phi_{\mu-\nu}(t)\, dt, \quad . \quad . \quad . \quad . \quad (117)$$

worin [72])

$$\left.\begin{aligned} \Phi_{\mu-\nu}(t) &= {}_2F_1\left(\tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu,\ \tfrac{1}{2} + \tfrac{1}{2}\mu - \tfrac{1}{2}\nu;\ \tfrac{1}{2};\ -t^2\right) \\ &+ (\mu-\nu)\, t\, .\, {}_2F_1\left(1 - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu,\ 1 + \tfrac{1}{2}\mu - \tfrac{1}{2}\nu;\ \tfrac{3}{2};\ -t^2\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (118a)$$

$$= 2^{\mu-\nu}\, t^{\mu-\nu-1}\, {}_2F_1\left(\tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu,\ 1 - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu;\ 1 - \mu + \nu;\ -t^{-2}\right). \quad (118b)$$

Die durch (112) definierte Funktion $G(z)$ ist daher gleich [73])

$$\int_0^\infty J_{4m}(4v)\, v\, dv \int_0^\infty J_{2m}\left(\frac{2v^2 t}{z^2}\right) \Phi_{2k}(t)\, dt$$

$$= \int_0^\infty \Phi_{2k}(t)\, dt \int_0^\infty J_{4m}(4v)\, J_{2m}\left(\frac{2v^2 t}{z^2}\right) v\, dv$$

$$= \tfrac{1}{4} z^2 \int_0^\infty J_{2m}\left(\frac{2z^2}{t}\right) \Phi_{2k}(t)\, \frac{dt}{t} \quad \text{(wegen (116))}$$

$$= \tfrac{1}{4} z^2 \int_0^\infty J_{2m}(u)\, \Phi_{2k}\left(\frac{2z^2}{u}\right) \frac{du}{u}$$

$$= \tfrac{1}{8} (2z)^{4k} \int_0^\infty J_{2m}(u)\, .\, {}_2F_1\left(\tfrac{1}{2} - k,\ 1 - k;\ 1 - 2k;\ -\frac{u^2}{4z^4}\right) u^{-2k}\, du$$

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71) MEIJER, [25], Formel (64).

72) Man beachte hierbei, dass

$$\frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\, z^a \cdot {}_2F_1(a,\ 1 + a - c;\ 1 + a - b;\ -z)$$

$$+ \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\, z^b \cdot {}_2F_1(b,\ 1 + b - c;\ 1 + b - a;\ -z) = {}_2F_1(a, b; c; -z^{-1})$$

ist (siehe BARNES, [5], 146). In [25] gebe ich (117) mit (118a); in der vorliegenden Arbeit brauche ich aber (118b).

73) Sind die Voraussetzungen (115) erfüllt, so ist die Vertauschung der Integrationsfolge erlaubt; man vergl. HOBSON, [16], 349.

wegen (118*b*). Das letzte Integral ist aber gleich [74])

$$\frac{\Gamma(\frac{1}{2}-k+m)}{8\,\Gamma(2m+1)}\,W_{k,m}(2z^2)\,M_{-k,m}(2z^2),$$

so dass der Beweis von (113), und also auch von (47), geliefert ist.

Beweis von Satz 11. Ich beweise zunächst

$$\left.\begin{aligned}&e^{m\pi i}\,W_{k,m}(te^{\frac{1}{2}\pi i})\,W_{-k,m}(te^{\frac{1}{2}\pi i})+e^{-m\pi i}\,W_{k,m}(te^{-\frac{1}{2}\pi i})\,W_{-k,m}(te^{-\frac{1}{2}\pi i})\\ &\frac{\Gamma(\frac{1}{2}-k+m)}{i\,\Gamma(2m+1)}\{e^{-k\pi i}\,W_{k,m}(te^{\frac{1}{2}\pi i})\,M_{-k,m}(te^{\frac{1}{2}\pi i})-e^{k\pi i}\,W_{k,m}(te^{-\frac{1}{2}\pi i})\,M_{-k,m}(te^{-\frac{1}{2}\pi i})\}.\end{aligned}\right\}\quad(119)$$

Man hat nämlich [75])

$$\left.\begin{aligned}W_{k,m}(z)\,W_{-k,m}(z)&=\frac{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}{2\pi i}\\ \times\{e^{-k\pi i}\,W_{k,m}(ze^{\pi i})\,W_{k,m}(z)&-e^{k\pi i}\,W_{k,m}(ze^{-\pi i})\,W_{k,m}(z)\}.\end{aligned}\right\}\quad(120)$$

Die linke Seite von (119) ist also gleich

$$\begin{aligned}&\frac{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}{2\pi i}\,[e^{-k\pi i}\,W_{k,m}(te^{\frac{1}{2}\pi i})\,\{e^{m\pi i}\,W_{k,m}(te^{3/2\pi i})+e^{-m\pi i}\,W_{k,m}(te^{-\frac{1}{2}\pi i})\}\\ &-e^{k\pi i}\,W_{k,m}(te^{-\frac{1}{2}\pi i})\,\{e^{m\pi i}\,W_{k,m}(te^{\frac{1}{2}\pi i})+e^{-m\pi i}\,W_{k,m}(te^{-3/2\pi i})\}]\\ &=\frac{\Gamma(\frac{1}{2}-k+m)}{i\,\Gamma(2m+1)}\,\{e^{-k\pi i}\,W_{k,m}(te^{\frac{1}{2}\pi i})\,M_{-k,m}(te^{\frac{1}{2}\pi i})-e^{k\pi i}\,W_{k,m}(te^{-\frac{1}{2}\pi i})\,M_{-k,m}(te^{-\frac{1}{2}\pi i})\}\end{aligned}$$

wegen (69), womit der Beweis von (119) geliefert ist.

Nun folgt aus (78) und (80), dass die rechte Seite von (48) existiert für $\Re(m)>-\frac{1}{4}$. Ich darf daher annehmen (analytische Fortsetzung), dass $\Re(m)>-\frac{1}{4}$ und $\Re(\frac{1}{2}-k+m)>0$ ist. Dann gilt wegen (47), mit $z=\zeta e^{\frac{1}{2}\pi i}$ $(-\frac{1}{2}\pi\leqq arg\,\zeta\leqq 0)$ und $u=ve^{-\frac{1}{2}\pi i}$ angewendet [76]),

$$\left.\begin{aligned}&\frac{\Gamma(\frac{1}{2}-k+m)}{i\,\Gamma(2m+1)}\,e^{-k\pi i}\,W_{k,m}(2\zeta^2e^{\frac{1}{2}\pi i})\,M_{-k,m}(2\zeta^2e^{\frac{1}{2}\pi i})\\ &\qquad=4\pi\zeta^2\int_0^{\infty e^{-i\,arg\,\zeta}}H^{(1)}_{m+k}(v^2)\,J_{m-k}(v^2)\,J_{4m}(4\zeta v)\,v\,dv.\end{aligned}\right\}\quad(121)$$

---

74) Man vergl. MEIJER, [26], Formel (18) (mit $\alpha=1+m$, $\beta=m$ und $2z^2$ statt z angewendet); siehe auch Formel (2) von [26].

75) Man findet (120), wenn man beide Seiten von (68) mit $W_{-k,m}(z)$ multipliziert und nachher k durch $-k$ ersetzt.

76) Man beachte auch, dass $I_\nu(we^{-\frac{1}{2}\pi i})=e^{-\frac{1}{2}\nu\pi i}J_\nu(w)$ ist (siehe (1)); ich verwende ferner (102).

In ähnlicher Weise folgt aus (47), mit $z = \zeta e^{-\frac{1}{4}\pi i}$ $(0 \leqq arg\,\zeta \leqq \frac{1}{2}\pi)$ und $u = v e^{\frac{1}{2}\pi i}$ angewendet [77]),

$$\left.\begin{aligned}\frac{\Gamma(\frac{1}{2}-k+m)}{i\,\Gamma(2m+1)}\, e^{k\pi i}\, W_{k,m}(2\zeta^2 e^{-\frac{1}{2}\pi i})\, M_{-k,m}(2\zeta^2 e^{-\frac{1}{2}\pi i}) \\ = -4\pi\zeta^2 \int_0^{\infty e^{-i\arg\zeta}} H^{(2)}_{m+k}(v^2)\, J_{m-k}(v^2)\, J_{4m}(4\zeta v)\, v\, dv.\end{aligned}\right\} \quad . \quad . \quad (122)$$

Aus (119), (121) und (122) geht nun hervor, falls $\zeta > 0$ ist,

$$e^{m\pi i}\, W_{k,m}(2\zeta^2 e^{\frac{1}{2}\pi i})\, W_{-k,m}(2\zeta^2 e^{\frac{1}{2}\pi i}) + e^{-m\pi i}\, W_{k,m}(2\zeta^2 e^{-\frac{1}{2}\pi i})\, W_{-k,m}(2\zeta^2 e^{-\frac{1}{2}\pi i})$$

$$= 4\pi\zeta^2 \int_0^{\infty} \{H^{(1)}_{m+k}(v^2) + H^{(2)}_{m+k}(v^2)\}\, J_{m-k}(v^2)\, J_{4m}(4\zeta v)\, v\, dv$$

$$= 8\pi\zeta^2 \int_0^{\infty} J_{m+k}(v^2)\, J_{m-k}(v^2)\, J_{4m}(4\zeta v)\, v\, dv,$$

womit Satz 11 bewiesen ist.

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77) $I_\nu(w e^{\frac{1}{2}\pi i}) = e^{\frac{1}{2}\nu\pi i} J_\nu(w)$ wegen (1) und (51); ich benutze weiter (101).

Mathematics. — *Zur Theorie der Kurven im* HILBERT*'schen Raum.* Von A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of September 24, 1938.)

Einleitung.

Es sei

$$x(t) = f(t; s) \quad (0 \leqq t \leqq 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

eine Kurve C im HILBERTschen Raum (d. h. im Raum, der im LEBESQUE'schen Sinne quadratisch-integrierbaren Funktionen von t); s sei die Bogenlänge dieser Kurve. Wir setzen voraus, dass die Ableitungen $\frac{dx}{ds}, \frac{d^2x}{ds^2}, \ldots$ alle existieren und linear unabhängig sind. Sind die Ableitungen linear abhängig, so liegt die Kurve bekanntlich in einem endlich viel-dimensionalen Unterraum des HILBERT'schen Raumes.

Weiter sei $f(t; 0) = 0$.

Mittels Orthogonalisierung der Vektoren $\dot{x}, \ddot{x}, \ldots$ erhält man in jedem Punkt der Kurve die Tangente und Normalen $L_0(t; s)$, $L_1(t; s) \ldots$ an die Kurve; diese Funktionen bilden also ein orthogonales normiertes System. Bekanntlich gelten dann die FRENET'schen Formeln:

$$\frac{dL_i}{ds} = \Gamma_{i-1}(s) L_{i-1} - \Gamma_i(s) L_{i+1} \qquad (i = 0, 1, \ldots; \Gamma_{-1} = 0.) \quad . \quad (2)$$

Die Koeffizienten Γ_i sind Funktionen von s und heissen die Krümmungen der Kurve im Punkt s; alle Ableitungen von Γ_i nach s existieren.

Im Nachfolgenden wird gezeigt, dass es im Sinne der Kongruenz eine und nur eine Kurve gibt mit vorgegebenen Γ_i unter Voraussetzung, dass die Γ_i analytische Funktionen sind in einer Umgebung von $s = 0$.

Aus diesem Satz wird dann weiter eine Eigenschaft abgeleitet über die Integrale eines gewissen simultanen Systems Differentialgleichungen mit unendlich vielen unbekannten Funktionen.

Für einige Ratschläge bin ich Herrn H. WEIJL in Princeton vielen Dank verschuldigt.

§ 1. Es gibt folgendes Theorem:

Theorem 1. Wenn für eine gegebene Kurve C gilt $|\Gamma_i(s)| \leqq M$, unabhängig von i und s, so liegen die Normalen und Tangenten für jeden Punkt der Kurve, also auch die Kurve selbst, ganz im linearen Raum, welcher durch die Tangente und Normalen im Ursprung $s = 0$ gebildet wird.

Beweis: Wie aus (2) folgt, gelten für

$$a_{ij}(s) = \int_0^1 L_i(t;s)\, L_j(t;0)\, dt. \quad \ldots\ldots\ldots \quad (3)$$

die Gleichungen:

$$\frac{da_{ij}}{ds} = \Gamma_{i-1}\, a_{i-1,j} - \Gamma_i\, a_{i+1,j}. \quad \ldots\ldots\ldots \quad (4)$$

Multiplizieren wir diese Gleichungen mit a_{ij} und summieren nachher über i, so erhält man

$$\frac{d}{ds} \sum_{i=0}^{n} a_{ij}^2 = -\Gamma_n\, a_{nj}\, a_{n+1\,j}. \quad \ldots\ldots\ldots \quad (5)$$

Da aber die a_{ij} in bezug auf das orthogonale System $L_i(t;s)$ Entwicklungskoeffizienten sind, so gilt $\lim_{n-\infty} a_{nj} = 0$, also folgt wegen $|\Gamma_n| \leqq M$:

$$\lim_{n\to\infty} \frac{d}{ds} \sum_{i=0}^{n} a_{ij}^2 = 0.$$

Weiter sind die Funktionen

$$\frac{d}{ds} \sum_{i=0}^{n} a_{ij}^2$$

gleichmässig beschränkt in bezug auf n und s. Mittels des LEBESQUE'schen Theorems über Limesübergang unter das Integralzeichen, folgt dann leicht

$$\sum_{i=0}^{\infty} a_{ij}^2(s) = \sum_{i=0}^{\infty} a_{ij}^2(0) = 1. \quad \ldots\ldots\ldots \quad (6)$$

Wegen (3) ist folglich

$$L_j(t;0) = \sum_{i=0}^{\infty} a_{ij}(s)\, L_i(t;s), \quad \ldots\ldots\ldots \quad (7)$$

im Sinne der mittleren Konvergenz. In diesem Resultat können die Werte 0 und s vertauscht werden sodass also die $L_i(t;s)$ im Raum der $L_i(t;0)$ liegen und ebenfalls die Kurve selbst:

$$x(s) = \int_0^s L_0(t;s)\, ds. \quad \ldots\ldots\ldots \quad (8)$$

Weiter beweisen wir das Existenztheorem:

Theorem 2. Es gibt eine und nur eine Kurve (im Sinne der Kongruenz) mit vorgegebenen Krümmungen $\Gamma_i(s)$ $(i = 0, 1, 2, \ldots)$ unter Annahme, dass

1^0. Es gibt eine Zahl $\varrho > 0$ derart, dass die $\Gamma_i(s)$ regulär-analytische Funktionen von s sind in der komplexen s-Ebene für $|s| \leqq \varrho$; ϱ ist unabhängig von i.

2^0. $|\Gamma_i(s)| \leqq M$ für $|s| \leqq \varrho$; M unabhängig von s und i.

Beweis:

I. Eindeutigkeit. Es seien C und C^* zwei Kurven mit Krümmungen $\Gamma_i(s)$; es kommen für diese beiden Kurven nur (reelle) Werte der Bogenlänge s in Betracht, für die $0 \leqq s \leqq \varrho$ ist.

$$C \ldots . x(t) = f(t; s).$$
$$C^* \ldots . x(t) = f^*(t; s).$$

Es gelten die Gleichungen (vgl. (4)):

$$\frac{da_{ij}}{ds} = \Gamma_{i-1}\, a_{i-1,j} - \Gamma_i\, a_{i+1,j}. \quad \ldots\ldots\ldots \quad (9)$$

$$\frac{da_{ij}}{ds} = \Gamma_{i-1}\, a^*_{i-1,j} - \Gamma_i\, a^*_{i+1,j}. \quad \ldots\ldots\ldots \quad (10)$$

Multiplizieren wir die Gleichungen (9) und (10) mit a^*_{ij} und a_{ij} und summieren nachher über i von 0 bis n, so kommt:

$$\frac{d}{ds} \sum_{i=0}^{n} a_{ij}\, a^*_{ij} = -\Gamma_n (a^*_{nj}\, a_{n+1,j} + a_{nj}\, a^*_{n+1,j}). \quad \ldots\ldots \quad (11)$$

Genau wie beim Beweise des Theorems I folgt hieraus wegen

$$a_{ij}(0) = a^*_{ij}(0) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \quad \ldots\ldots\ldots \quad (12)$$

dass

$$\sum_{i=0}^{\infty} a_{ij}\, a^*_{ij} = 1. \quad \ldots\ldots\ldots \quad (13)$$

Aus der, jetzt gültigen, Formel (7) folgt noch

$$\sum_{i=0}^{\infty} a^2_{ij}(s) = \int_0^1 L_j^2(t; 0)\, dt = 1 \quad \ldots\ldots\ldots \quad (14)$$

$$\sum_{j=0}^{\infty} a^2_{ij}(s) = \int_0^1 L_i^2(t; s)\, dt = 1 \quad \ldots\ldots\ldots \quad (15)$$

und die analogen Gleichungen für die Kurve C^*. In Verbindung mit (13) folgt jetzt leicht

$$a_{ij}(s) = a^*_{ij}(s). \quad \ldots\ldots\ldots \quad (16)$$

Es haben also die Normalen von C und C^* für $0 \leqq s \leqq \varrho$ dieselben Koordinaten, jede in bezug auf das zugehörige Koordinatensystem, d. h. in bezug auf die Systeme $L_j(t;0)$ und $L^*_j(t;0)$. Für $i=0$ gilt speziell:

$$f'(t;s) = \sum_{j=0}^{\infty} \alpha_j(s)\, L_j(t;0)$$

$$f^{*\prime}(t;s) = \sum_{j=0}^{\infty} \alpha_j(s)\, L^*_j(t;0)$$

folglich

$$\left.\begin{aligned} f(t;s) &= \sum_{j=0}^{\infty} \int_0^s \alpha_j(s)\, ds \,.\, L_j(t;0) \\ f^*(t;s) &= \sum_{j=0}^{\infty} \int_0^s \alpha_j(s)\, ds \,.\, L^*_j(t;0) \end{aligned}\right\} \quad \ldots\ldots \quad (17)$$

Zufolge des Satzes, dass eine Funktion (abgesehen von Nullmengen) eindeutig durch ihre Entwicklungskoeffizienten in bezug auf ein orthogonales normiertes System bestimmt ist (mag dieses vollständig sein oder nicht), kann das Resultat auch in folgender Weise ausgesprochen werden:

Wird C^* eine neue Kurve C^{**} zugeordnet, derart, dass C^{**} in bezug auf das System $L_i(t;0)$ dieselben Koordinaten hat wie C^* in bezug auf $L^*_i(t;0)$, so ist

$$C^{**} \equiv C.$$

II. Existenz. Wie aus dem Eindeutigkeitsbeweise deutlich hervorgeht, wird die Kurve, falls sie überhaupt existiert, eindeutig bestimmt sein nach Annahme der Tangente und die Normalen in einem Punkte der Kurve, z. B. $s=0$. Es wäre also zu zeigen, dass es eine Kurve gibt mit Krümmungen $\Gamma_i(s)$ und für welche

$$L_i(t;0) = \varphi_i(t). \quad \ldots\ldots\ldots \quad (18)$$

Hierin bedeutet $\{\varphi_i(t)\}$ ein vorgegebenes orthogonales normiertes Funktionensystem, von dem wir voraussetzen wollen, dass die $\varphi_i(t)$ endlich sind für $0 \leqq t \leqq 1$. Das Problem ist so zurückgeführt auf die Integration des simultanen Systems (2) mit Anfangsbedingungen (18). Mittels Differentiationen der Gleichungen (2) nach s, lassen sich sofort die Ableitungen $\frac{d^k L_i}{ds^k}$ $(i=0, 1, 2, 3, \ldots;\ k=1, 2, 3\ldots)$ für $s=0$ ausdrücken in die $\varphi_j(t)$, die $\Gamma_j(0)$ und die Ableitungen der Γ_j in $s=0$.

Die Existenz der Lösungen $L_i(t; s)$ wäre somit sichergestellt, wenn nur noch die Konvergenz der betreffenden TAYLOR-reihen bewiesen wäre. Man sieht aber sofort ein, dass das Letztere gelingt mittels der Majorantenmethode CAUCHY's („Calcul des limites"), genau so, wie das bei einem System mit endlich vielen Unbekannten geschieht. Die $L_i(t; s)$ werden also in der komplexen s-Ebene regulär-analytische Funktionen für $|s| < \lambda$; sind die Γ_i ganze Funktionen, so werden das auch die L_i [1]). Die gesuchte Kurve wird jetzt sein:

$$x(t) = \int_0^{\cdot} L_0(t; s)\, ds \quad (s < \lambda) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Wir zeigen dazu:

$$1^0. \qquad \int_0^1 L_i(t; s)\, L_j(t; s)\, dt = (L_i, L_j) = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \quad . \quad . \quad . \quad (20)$$

Das gelingt mittels vollständiger Induktion. Aus den Gleichungen (2) folgt, da die φ_i ein orthogonales normiertes System bilden

$$\left[\frac{d}{ds}(L_i, L_j)\right]_{s=0} = 0.$$

Weiter zeigt sich, dass

$$\psi_k(s) \equiv \frac{d^k}{ds^k}(L_i, L_j)$$

sich linear ausdrücken lässt in einer endlichen Anzahl der (L_λ, L_μ). Eine einfache Ueberlegung lehrt dann, dass aus $\psi_k(0) = 0$ folgt $\psi_{k+1}(0) = 0$. Da (L_i, L_j) eine analytische Funktion ist, ist also

$$(L_i, L_j) = \text{Konstant} = \delta_{ij}.$$

2^0. Es seien $\Gamma_i^*(s)$ die Krümmungen und $L_i^*(t; s)$ die Tangente und Normalen der Kurve (19). Dann ist also zu zeigen $\Gamma_i^* = \Gamma_i$; $L_i^* = L_i$. Auch das gelingt mittels vollständiger Induktion. Man hat:

$$\frac{dL_i}{ds} = \Gamma_{i-1} L_{i-1} - \Gamma_i L_{i+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\frac{dL_i^*}{ds} = \Gamma_{i-1}^* L_{i-1}^* - \Gamma_i^* L_{i+1}^* \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

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[1]) Für die Kurve kommen nur reelle Werte $0 < s < \lambda$ in Betracht.

Es sei

$$\left.\begin{aligned} L_j^* &= L_j \text{ für } j \leqq i \\ \Gamma_j^* &= \Gamma_j \text{ für } j \leqq i-1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (23)$$

Wegen (20) ist dann (auch die L_i^* bilden ein orthogonales normiertes System):

$$\Gamma_i = \left\{ \int_0^1 (\Gamma_{i-1} L_{i-1} - \dot{L}_i)^2 \, dt \right\}^{\frac{1}{2}}$$

$$\Gamma_i^* = \left\{ \int_0^1 (\Gamma_{i-1}^* L_{i-1}^* - \dot{L}_i^*)^2 \, dt \right\}^{\frac{1}{2}}.$$

Mit (23) ist also $\Gamma_i = \Gamma_i^*$, und dann auch $L_{i+1} = L_{i+1}^*$.

Für $i = 0$ sind die Gleichungen (23) aber gültig, so dass allgemein $\Gamma_i = \Gamma_i^*$.

Das Theorem 2 ist somit vollständig bewiesen.

§ 2. Aus obenstehenden Beweisen folgt unmittelbar eine merkwürdige Eigenschaft der Integrale des Systems

$$\frac{dx_i}{ds} = \Gamma_{i-1} x_{i-1} - \Gamma_i x_{i+1}. \quad \ldots\ldots\ldots \quad (24)$$

$$(i = 0, 1, 2, \ldots; \Gamma_{-1} = 0).$$

Es sei $\{a_{ij}(s)\}$ $(i = 0, 1, \ldots)$ das für $|s| < \lambda$ sicher existierende Lösungssystem, für welches $a_{ij}(0) = \delta_{ij}$ ist. Es gelten dann die Reihenentwicklungen

$$\left.\begin{aligned} \sum_{i=0}^{\infty} a_{ij}^2(s) &= 1 \quad (j = 0, 1, \ldots) \\ \sum_{j=0}^{\infty} a_{ij}^2(s) &= 1 \quad (i = 0, 1, \ldots) \end{aligned}\right\} \quad \ldots\ldots \quad (25)$$

Diese Reihen sind Verallgemeinerungen der HANSEN'schen Reihe aus der Theorie der BESSELschen Funktionen

$$J_0^2 + 2J_1^2 + 2J_2^2 + \ldots = 1. \quad \ldots\ldots\ldots \quad (26)$$

Setzt man nämlich $\Gamma_0 = \frac{1}{2}\sqrt{2}$, $\Gamma_i = \frac{1}{2}$ $(i \neq 0)$, so findet man leicht

$$a_{ij} = \varepsilon_{ij} \{J_{i-j} + (-1)^j J_{i+j}\}. \quad \ldots\ldots\ldots \quad (27)$$

wenn noch:

$$\varepsilon_{00} = \tfrac{1}{2}$$

$$\varepsilon_{i0} = \varepsilon_{0j} = \tfrac{1}{2}\sqrt{2} \quad (i \neq 0,\ j \neq 0)$$

$$\varepsilon_{ij} = 1 \qquad (i \neq 0,\ j \neq 0).$$

Bekannte Formeln führen dann von den Gleichungen (25) zur Reihe (26).

Bemerkungen: 1. Sind die Γ_i ganze Funktionen, so sind das auch die Lösungen a_{ij}.

2. Für die Beweise der in der Einleitung genannten Sätze sei verwiesen nach:

S. MINETTI, Sur quelques espaces fonctionnels etc. Mém. des Sc. mathématiques fasc. 69 (1936).

A. F. MONNA, Krommen in een functieruimte. Mathematica (Thieme) VI u. f. (1937—1938).

Botany. — *Some chemical properties of the plastid-granum.* By W. F. H. M. MOMMAERTS, (from the Botanical Institute, Government University, Leyden). (Communicated by Prof. L. G. M. BAAS BECKING.)

(Communicated at the meeting of September 24, 1938.)

§ 1.

In recent years it has become increasingly evident that the chlorophyll in the living leaf is present as a prosthetic group of a proteid. The publications of NOACK, ARNOLD and, in particular, the work done at the Leyden Botanical Laboratory have supplied arguments in favour of this opinion (1). The analogy with the chemistry both of enzymes and of respiratory pigments is apparent. The existence of an "agon-pheron"-unit is believed to be essential to the mechanism of photosynthesis (2). It seems that the chlorophyll-proteid is combined with both carotinoids and lecithinoids into a "symplex" in the sense of WILLSTÄTTER and that, in the granum (3) this symplex is present in a regular pattern (4). The name phyllochlorin, as used by MESTRE (5) to designate this symplex has precedence over STOLL's chloroplastin (6) and will therefore be used in this paper to designate this complex structure.

§ 2.

Attempts were made to isolate this phyllochlorin. According to NOACK and also to LUBIMENKO (7) grinding of leaves in tapwater yields a green suspension containing nuclear and protoplasmic material, cell-wall fragments, intact- and broken plastids, but also green particles of a much smaller dimension than that of the chloroplast. It is believed that these particles constitute the grana (8). By means of fractional centrifugation these grana may be separated from the rest of the cell-fragments; at lower speeds these other fragments are thrown down while the grana are separated from the aqueous phase by high-speed centrifugation. MENCKE (9) who used this method to determine the chemical composition of various cell-constituents, does not recognize the grana as separate entities, as he speaks of "Chloroplastensubstanz".

The reality of the grana, however, seems to us beyond dispute, as will be demonstrated below.

Method of preparation. Washed leaves of sweet pea, clover, spinach, nettle and *Nicotiana macrophylla* are ground in tapwater to which calcium carbonate was added, under continuous cooling with ice. The extract is filtered through cloth, the residue being used again. The dark green suspension is centrifuged for about 30 minutes at 3500—4000 rev./min.

in order to throw down most of the cell fragments. Part of the grana are also sedimented in this way. The remaining liquid is centrifuged for 90 minutes. The sediment is collected and stirred into distilled water, after which it is centrifuged again. This procedure is repeated twice. Impurities appear as a whitish zone in the sediment. This stratum is removed. The effect of each manipulation is controlled microscopically. The green mass obtained in this way consists of grana, which are stored in the ice-chest.

The size of the particles and also the fact that grana become more clearly visible on injury of the plastid gives additional support that the sediment actually consists of structural units. The following observation appeared to us particularly significant. When a small fragment of a leaf is torn with needles, high magnification shows broken as well as intact plastids. The liberated grana seem in all respects identical with those still enclosed within the plastid. They possess the characteristic flat shape. Sometimes a broken chloroplast may be seen with a half-protruding granum. From time to time such a granum separates itself from the plastid by its Brownian movement, it enters the liquid phase and is indistinguishable from other green particles in the liquid.

In all plants examined practically all of the plastids disintegrate during the preparation of the suspensions.

It is hardly likely, however, that the grana remain undisturbed during the treatment, as progressive swelling may often be observed. Therefore hydratation may take place and possibly extraneous material may be adsorbed upon the granum-surface. Both factors may cause uncertainties in the determination of the chemical composition of the granum. For the present investigations they seem of secondary importance.

§ 3.

Suspending the grana in alcohol or acetone ($\pm$ 40 %), or ammonium sulphate (half-saturated) causes them to lose their structure, and to form an amorphous precipitate, this precipitate being the phyllochlorin in its unorganized form, which is but little soluble, in agreement with the hydrophobic character of most of its constituents [1]).

The sediment is centrifuged, and washed twice with cold distilled water. For chlorophyll- and protein determinations the sediment is suspended in cooled acetone (85 %), centrifuged, and the residue extracted with pure acetone. The pigments are completely removed in this way. The remnant is extracted twice with ether, to dissolve the remaining lipoids out of the protein. The residue shows protein reactions; it is dried over paraffin wax (and, if desired, over phosphoric anhydride) and weighed [2]). The

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[1]) This experiment is inspired by the work of LUBIMENKO (loc. cit.); one cannot say, however, that L. isolated the phyllochlorin because his "chlorophylle naturelle" contained nearly the whole cell.

[2]) Because the N. percentage is not yet known, this procedure gives more reliable results than f.i. micro-Kjeldahl.

chlorophyll-content is estimated by determining the absorbtion coefficient for red light (ruby glass) with a Selenium-cell. For comparison pure chlorophyll ("97 %", according to STOLL) was used; it was kindly put at my disposal by E. A. HANSON, M.A. According to these determinations, the protein-chlorophyll ratio was found to be about 100 : 5.5. *On the assumption, that one protein-unit with a particle weight of 17.000* (SVEDBERG), *carries one chlorophyll molecule (average M.W. 926), one hundred parts of protein should carry 5.45 parts of chlorophyll.* We may conclude, therefore, that the assumption of an agon-pheron compound is correct. It is interesting to note that the molecular ratio protein: prosthetic groups is the same as in other conjugated proteins of biochemical importance, f.i. haemoglobin, cytochrome *c*, and others.

It should be mentioned that during the experiments a serious difficulty presented itself. Particularly in the experiments with nettles, it was observed that grinding of the leaves at a lower pH (SØRENSEN's phosphate buffer pH 6.8)led to irregular results (ratio's of 100 : 10 or more), while extracting with water and calcium carbonate gave results equal to, or close to, the theoretical value. (With nettles exact results are only attainable under special precautions). This may be due to a process of autolysis of the protein, or to different mode of separation of the proteins, caused bij a change in the electric charges. To investigate these possibilities, the following experiment was carried out; leaves were ground with water, calcium carbonate and a trace of cupric chloride, and the grana were purified in the usual way. A second lot of leaves was ground with SØRENSEN's buffer, pH = 6.8; the solid constituents of the suspension were removed and the remaining brownish liquid was mixed with the grana, isolated from the first lot of leaves. The resulting suspension was divided into two portions; of one portion (A), the grana were purified, all the manipulations being carried out as quickly as possible. The other portion (B) was allowed to stand for some hours, and then treated similarly. The results of the determinations of the protein-chlorophyll ratio's were for (A); 100 : 5.5, for (B) 100 : 6.3. This experiment clearly shows that the second supposition (direct "colloid-chemical" influence of the pH) does seem to apply. On the other hand, supposing the protein to be partly autolysed, the difference between (A) and (B) might be expected to be higher; however, it is easily conceivable that the proteolytic agent is mainly removed together with the grana and other cell constituents, and is therefore strongly diminished in fraction (B).

The progressive autolytic action, even at an unfavourable pH, is better demonstrated in another experiment: leaves were ground in SØRENSEN's phosphate buffer (pH = 8.0). From one portion the grana were purified immediately, carrying out all the manipulations at low temperature and as quickly as possible. From the other portion the grana were purified after a few hours standing. The protein-chlorophyll ratio's were found to be 100 : 5.7, resp. 100 : 7.5.

From the foregoing experiments it may be concluded that the assumption of autolysis is justified; the fact that the phenomenon is inhibited by traces of cupric ions, as is autolysis in yeast and in animal tissues, is also in favour of this view.

§ 4.

The question arises whether the quantitative relation between protein and chlorophyll, as demonstrated above, finds support in earlier literature.

1. VON EULER, BERGMANN and HELLSTRÖM (10) found that one Elodea-chloroplast of 40 μ^3 contained 2.75—10^{-15} gr. mol. or 16.67×10^8 molecules of chlorophyll. At that time, the granular structure was not yet rediscovered; the Elodea chloroplast is densely filled with small grana. Now, one molecule of a conjugated protein, with a molecular weight of 68.000 (inclusive water and prostethic group) has the size of about $40 \times 40 \times 50$ Å (L. W. JANSSEN, oral communication; in good agreement with the experimental results of ZEILE, KUNITZ and ELFORD). In our case, the protein c.s. would take a volume of:

$$\frac{16.67}{4} \times 10^8 \times 40 \times 40 \times 50 \text{ Å}^3,$$

or about 30 μ^3, which is of the right order of magnitude.

2. E. A. HANSON (loc. cit.) measured the chlorophyll content of a sample of Hormidium cells and estimated the number of grana in the sample and their approximate dimensions (exact measurement is not possible). By a calculation similar to that employed above, he found a close agreement between the granular volume and the chlorophyll-protein volume.

3. HANSON, MEEUSE, MOMMAERTS and BAAS BECKING (11) determined the chlorophyll-content of the granum indirectly comparing the extinction of red light by the granum (H. D. VERDAM, unpublished), with that of stratified chlorophyll layers, obtained with the Blodgett-technique (12). They found either $\pm$ 5 % or 10 %, due to a methodical uncertainty in VERDAM's experiment.

The four different series of measurements, therefore, seem to confirm each other very satisfactorily.

§ 5.

Recently FRENCH (13) prepared a bacteriochlorophyll-carotinoid-protein compound from purple bacteria. It seemed promising to take up the quantitative investigation of this compound.

Bacteria (Rhodospirillum palustre MOLISCH) were centrifuged from the culture liquid. Having no device for high-frequency vibration at my disposal, I desintegrated the bacterial cells by plasmolysis with ice-cold

ammonium-sulphate (half saturated), by which also the chromoproteid and other substances are precipitated. The sediment was washed with CLARK and LUBS' buffer, pH = 4.0. According to FRENCH, the chromoproteid is insoluble at this pH, but is soluble at other degrees of acidity. For pH = 7.2 (CLARK and LUBS' buffer or SØRENSEN's phosphate buffer) I could not confirm this fact; at this pH the degree of dispersity is apparently higher, so that the sedimentation constant in the centrifugal field is lower than at pH 4.0, thus suggesting solubility. However, this difference with FRENCH's result may be due to the different mode of preparation. Making use of this higher dispersity at neutral reaction, I tried a further purification, which seemed to be ineffective.

From this product the protein-bacteriochlorophyll ratio was determined. The protein was estimated gravimetrically, the bacteriochlorophyll on its magnesiumcontent which was determined by means of "Titan-yellow" (KOLTHOFF, 14). Finding 110 mgr. protein, $\frac{24.3}{17.000} \times 110$ mgr. or 157 γ magnesium should be expected. The result was much lower (less than one-half); in agreement with the assumption that the chromoproteid was very impure. Further purification and accurate determinations will be attempted in the near future.

§ 6.

From a study of the action of HCN, WARBURG (15) concluded that "Schwermetallkatalyse" played a part in photosynthesis or more precisely, *an iron-porphyrin compound catalyses the dark reaction.* Although sensitivity to HCN is also conceivable without heavy metal (16), the action of other substances upon the dark reactions supports the idea. Iron determinations in the grana should be, therefore, not without interest.

Grana suspensions were prepared in the usual way; they were placed in SCHLEICHER and SCHÜLL's dialyzing tubes for two weeks at low temperature. In the dialyzate no iron was detectable with chemical means. A portion was dried, and incinerated in a platinum crucible with some Na_2CO_3 and KNO_3. A second portion was precipitated with acetone 40 %, centrifuged and treated in the same manner. After incineration the residues were extracted with 2n HCl and brought to a volume of 10 cc. Iron determinations were performed by means of ammonium thiocyanate and concentrated by means of ethylacetate, magnesium determinations (for the chlorophyll content) after neutralization, was performed by means of Titan-yellow and 4 N NaOH.

The grana, prepared in this way, contain iron, apparently in an organic state. There seems to be one iron atom to several tens of chlorophyll molecules. The nature of the iron compound will be investigated further [1]).

[1]) Some provisional experiments showed that the grana suspensions possess a weak catalase action. Many iron-porphyrin compounds have this property which I believe to be of no interest to the mechanism of photosynthesis, because the inequality of the catalase-theory has been definitely shown (20).

§ 7.

From the foregoing it is seen that it is possible to isolate a compound of protein, chlorophyll, carotinoids and probably lipoids. The nature of the mutual attraction of the components will be briefly discussed.

Concerning the chlorophyll, a protein-bound state is very probable because of the exact stoechiometric relations and the spectral properties (HUBERT, loc. cit.).

As to the carotinoids, the answer is more difficult. Without giving a definite opinion at this place, I will mention some facts which seem to oppose the assumption of a protein-bound state of the carotinoids:

1. If the carotinoids were bound to the same protein as the chlorophyll, there should be a definite relation between the number of chlorophyll molecules and the number of carotinoid molecules. This is not the case. For a protein carrier, apart from the carotinoids, there is "no room" (§ 4): moreover such a protein was not detected during the present investigation.

2. If the carotinoids were bound to a protein, very considerable alteration of their spectral properties should be detectable (compare the data about the astacin (17) and the visual pigments (18)). A systematic study of the leaf carotinoids on this point, comparable with HUBERT's work on the chlorophyll, does not exist. From the data available, however, it appears that such a band-shift does not exist.

3. Photochemical function correlates with protein-bound state: chlorophyll, phycoerythrin and phycocyanin. Also the photoreceptoric carotinoids in the retina are protein-bound. The fact that the plastid carotinoids have no photochemical function in photosynthesis, fits therefore well with the assumption that they are not bound to a protein.

The most probable conclusion is, therefore, that the chlorophyll and the protein form a stoichiometrical compound and that the carotinoids, and probably also the other lipoids are kept together in the system by (homoiopolar) cohesion-forces. The lipophilic nature of the symplex explains its insolubility. It may be expected that a higher dispersity of the system may be obtained with the aid of suitable chemicals. This gives the explanation of the fact that, according to a recent publication, SMITH (19) obtained "solutions" of the phyllochlorin with digitalin or bile-salts which solutions show no visible particles. It will be clear that such systems are wholly artificial and that the "molecular weights" of its components have neither chemical nor physiological significance, except when the reality of these compounds is proved by other means [1]).

As an argument for the combined state of the pigments, it is often stated that for the extraction of the pigments from the leaf, a certain amount of water is necessary which should "hydrolyse" the compound. From carefully dried grana, the pigments are easily extracted with dry acetone,

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[1]) Moreover, SMITH's phyllochlorin is certainly not pure. The material which is extracted also contains nuclei, cell-wall fragments, and so on (see § 2).

ether, benzene, petroleum-ether (low-boiling), CS_2 and CCl_4. The phenomenon is therefore not real and seems to be caused by the protective action of the surrounding cell structures.

SUMMARY.

1. The significance of the protein-bound state of chlorophyll is briefly mentioned.

2. A method is described for the preparation of grana-suspensions, with additional evidence about the identity of the isolated material.

3. From such a grana-suspension, the phyllochlorin may be obtained in an unorganized form.

4. In the phyllochlorin the ratio protein-units to porphin-nuclei is 1 : 1, as is the case in other conjugated proteins of high biochemical importance.

5. A comparison is made between these results and other data about the chlorophyll content of the photosynthetic system. These data show satisfactory agreement.

6. In the case of the purple-bacteria, the preparation of the symplex did not yet succeed.

7. The grana-preparations contain iron in a bound state which is of great interest in relation to the so-called "dark-reaction". They show a weak catalase-action which may be due to this compound.

8. It is probable that chlorophyll is chemically bound to the protein but that the carotinoids are only loosely attached by cohesion-forces. The protein-bound state seems to be essential for photochemical action.

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Botany. — *Some properties of Chlorophyll-multifilms.* By M. F. E. NICOLAI and C. WEURMAN, (from the Botanical Institute, Government University of Leyden). (Communicated by Prof. L. G. M. BAAS BECKING).

(Communicated at the meeting of September 24, 1938.)

In earlier work from the Botanical Laboratory (HANSON, MEEUSE and MOMMAERTS) (4) the technique as developed by LANGMUIR and BLODGETT for the preparation of multifilms was applied to chlorophyll.

These experiments were continued and extended by the authors of this paper. According to the model of the photosynthetic unit as devised in this laboratory by HUBERT (6) chlorophyll monolayers should alternate, at the hydrophilic side with protein — and at the hydrophobic side with lecithin films. It seemed interesting, therefore, to study the properties of the two- or three-component multifilm.

As the carrier-protein of the chlorophyll is as yet insufficiently known and also difficult to obtain, we used globin, prepared from ox-blood (HILL and HOLDEN) (5), instead. It is not unreasonable to suppose that this protein which, in blood, is linked to haemin, will be related to the chlorophyll-carrier. Moreover, NOACK (8) succeeded in preparing fluorescent chlorophyll-globin adsorbates. As also the lecithinoids of the granum are of unknown composition, we used the commercial preparation "planticin" and "ovolecithin", purified according to the methods developed by BUNGENBERG DE JONG and coworkers (3). Multifilms were either built on polished glass or on polished metal slides, which slides were previously covered with a number of Barium-stearate monofilms.

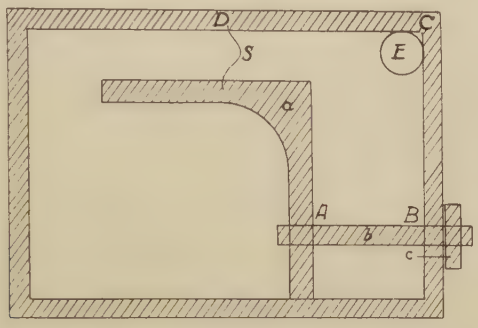
Chlorophyll, lecithin or globin was spread on a BLODGETT-trough, the boundary of the film being formed by a paraffined silk thread. Outside the thread a constant counterpressure of 15 dynes/cm^2 was exerted by means of "piston-oil", in our case castor-oil.

Dipping or raising the slide through the monolayers causes a contraction of the boundary of the silk thread. By carefully tracing the contour of the thread on transparent paper before and after each plunge it appeared that the amount of film taken up by the slide by dipping it into the liquid ("A" layers of LANGMUIR) was always smaller than the area taken up by the slide by raising it through the film ("B" layers of LANGMUIR).

Exact measurements were needed to verify this impression. As our troughs were too large (37 × 57 cm) to allow of the use of a floating barrier between piston oil and spreading substance to measure the uptake (LANGMUIR) (7*a*) and as, moreover, alternating layers of chlorophyll and protein (or lecithin) had to be measured, we had to resort to another

procedure. A horizontal projection of the alterations made is given in the accompanying figure.

A paraffined, L-shaped piece of glass (a) is placed in the trough at a



distance of 12—14 cm from the right hand side. A silk thread (S) separates the film to be investigated from the piston-oil surface.

A sliding glass bar (b) may move over the piece (a) and the side of the trough and will, when moved, compress the film confined between moveable bar and silk thread (S). The needle c indicates the position of the bar on a scale pasted on the lower surface of the bottom of the trough. After the substance to be investigated is spread on the area A B C D and piston-oil is applied outside, the position of the thread is carefully traced on transparent paper, pasted on the bottom of the trough. The slide, when dipped into the vessel E, causes the thread S to move to the right, the thread being brought back to its original position by moving the bar (b).

Multiplication of the distance travelled by the bar (b) and the distance A B gives the area of the monolayer taken up by the slide.

Using the above-mentioned method to determine the amount of the surface taken up by the plate, striking differences were found between the uptake of A- and B-layers; also the uptake was influenced by the substrate upon which the films were deposited. Those differences will show clearly in the description of some experiments.

1. A so-called "X-film" (according to the terminology of BLODGETT (2) a film of one kind of mono-layers, A or B) was built, transferred from a chlorophyll-surface spread on tapwater (pH 7,7) to an object glass which had been covered with 7 layers of globin (globin has proved to be even better than stearate to act as substrate for the chlorophyll films).

Length of the moving bar: 12 cm.

Surface of the slide covered by the film: 26.5 cm^2.

On building A-films, the following displacements of the bar (in cm) were measured:

A. 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1.95	2.00	2.00	2.15	2.05	2.20	1.90	1.95	2.15	2 10	2.10	2.00	2.10	2.20	2.00

From those data results an average uptake of 2.06 × 12 cm^2 = 24.7 (22.8 — 26.4) cm^2 pro A-layer.

The surface of the slide being 26.5 cm^2. we find that the slide on dipping takes up 93 (84 — 100) % of its surface.

2. Another X-film consisting of B-layers was built under the same circumstances as mentioned above for the A-layers.

Length of the moving bar: 12 cm.

Surface of slide covered by the film: 24.4 cm^2.

Subsequent displacements of the bar:

B. 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2.55	2.55	2.40	2.55	2.65	2.40	2.50	3.65	2.30	3.40	2.40	2.55	2.60	2.00	2.25	2.50	2.40	2.70	2.50

The average movement: 2.57 cm amounts to an average uptake of 2.57 × 12 = 30.8 cm^2 (limits 24 and 43.8 cm^2) corresponding to 126 % (100 — 179 %) of the slide surface. In practically all instances the area taken from the B-layer was larger than that taken up from the A-layer.

3. "Y-films" (alternating A- and B-layers) yielded similar results. We found as averages:

	Film 1	Film 2 [1])
Uptake A-layer	97 %	95.5 %
Uptake B-layer	104 %	106 %

4. "Y-films" built up from chlorophyll A—B, alternating with globin A—B, the A-chlorophyll being deposited upon the B-globin and vice versa, showed similar effects. Here the uptake of the A-layer was 84.5-, of the B-layer 101.5 % of the slide-surface.

The difference between A- and B-layer was equally apparent on metal slides.

The multifilms spread on glass yielded preparations suitable for extinction-measurements. These measurements were carried out by means of a selenium cell, and light filtered through ruby-glass.

The results obtained with various preparations are given below.

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[1]) Here the subsequent movements of the bar not only illustrate the difference in uptake between A- and B-films, but also the higher variability in the B-values.

Layer	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	2.40	2.20	2.05	2.00	2.00	2.25	2.15	2.20	2.00	2.00	2.15	2.10	2.15	2.10	2.20
B	2.70	2.25	2.45	2.25	2.30	2.35	2.35	2.30	2.25	2.55	2.35	2.55	2.25	2.30	2.25

$I = I_0\ e^{-kn},\ I_0 = 100$

	n number of layers	I transmitted	$I_0 - I$ absorbed	$k\ log\ e$	% Surface taken up pro layer
(AB) chlor.	26 (× 2)	73.6	26.4	0.00254	$\frac{96+105}{2}$
(AB) chlor. on globin	24 (× 2)	78.9	21.1	0.00215	$\frac{84.5+101.5}{2}$
A chlor.	32	89.2	10.8	0.00153	93
B chlor.	38	78.9	21.1	0.00271	126
B chlor.	64	66.0	34.0	0.00282	126

From this table it appears that there is a distinct difference in extinction between XA-, XB- and Y-films, as well as a difference between an Y-film of chlorophyll alternating with globin and an Y-film of only chlorophyll.

Further data are needed in order to verify whether a correlation exists between the amount of the surface taken up pro layer and the extinction pro layer.

The extinction coefficient for one molecule (for red light) appears to be in the neighbourhood of $\frac{0.00235}{log\,.\,e} = 0.005$.

Experiments with the other possible components of the photo-synthetic unit; the lecithinoids, yielded unexpected results.

For a lecithin-film could not be composed of alternating A- and B-layers the B-layer having a pronounced tendency to slip off the slide entirely on the next dipping. This stripping could neither be prevented by changes in pH nor by the so-called "conditioning" of the slide-surface with aluminum-chloride. Only two combinations seemed to form stable films.

1. A large number of layers could be stacked when lecithin B-layers were alternated with chlorophyll B-layers.

2. The combination lecithin A — chlorophyll B also yielded a permanent film.

None of the combinations prepared by us showed fluorescence. From the percentage of "uptake" of the slides by chlorophyll-films some conclusions may be drawn. HANSON and coworkers (4) found that the chlorophyll molecules when spread in a monolayer cover each other like tiles, the planes of the flat porphine nuclei being tilted under an angle of about 55° with the water surface.

Inasmuch as the A-covering of our slides differed only slightly from 100 % it may be assumed that also on these slides the molecules are stacked in a similar manner, the stacking of the B-layer may be slightly steeper ($> 55°$) than that of the A-layers. Now the uptake from A-layers

of chlorophyll on a globin film is markedly decreased (uptake only 84 %). Still this decrease is not sufficient to assume a flat position of the chlorophyll molecule in this case. For if the molecules formed a closed, flat pattern upon the globin they need only a little more than 25 % of the equivalent area. It may be that in the fluorescent globin-adsorbates as obtained by NOACK the chlorophyll actually formed such a pattern. In order to see whether the failure to obtain such (fluorescent) adsorbates might be due to a change in state of the protein a globin film composed of 400 monolayers was examined under polarized light as GORTER has recently demonstrated (1) that protein films consisting of 1700 monolayers exhibited distinct birefringence. No such preferred direction was found in the specimen prepared by us, neither did a (deep green) film built from 180 chlorophyll monolayers show measurable birefringence. However the number of layers may be far too small to obtain results.

Professor I. LANGMUIR called our attention to a possible source of error in our method resulting from the non-uniform distribution of the pressure in our films, which heterogeneity may be caused by the shape of the surface. The method described above is therefore only strictly applicable to liquid films. As in our subsequent experiments the shape of the surface was never alike, it seems that the method may also be suitable for non-liquid films.

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Geology. — *An Early Palaeolithic Site on the Northern Veluwe.* By F. C. BURSCH, F. FLORSCHÜTZ and I. M. VAN DER VLERK. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of September 24, 1938.)

It is several months ago that Major J. MALLINCKRODT, of the Infantry-School for aspirant-reservist-officers at Kampen, found a piece of flint, bearing obvious traces of human workmanship, upon a newly made bicycle-path in the northern part of the Veluwe. Although at first he did not intend to make this single discovery public, when during last summer he found a number more worked flints in the heaps of loam beside various paths in the neighbourhood, the case became entirely different.

Not satisfied by the fact that his archaeological acumen enabled him soon to collect a large number of artefacts, he pursued his research further and did not rest until he had traced the origin of the heaps of loam amongst which they had been discovered.

Thus he found in the wall of a loam-pit in a special, rather narrow layer, a very large quantity of implements as well as half-finished and waste pieces that always accompany them *in situ*.

Being of opinion that he ought not keep this discovery to himself and convinced that his find was of sufficient importance to form a valuable contribution to our knowledge of the earlier inhabitants of our country, he immediately informed the authorities of the National Archaeological Museum at Leiden, at the same time, with great generosity, offering all his material for the collections of this institution.

Thanks to the cordial and unselfish co-operation of the discoverer, the writers of this article were allowed to inspect the finding-place and the artefacts and so were enabled, as on a former occasion, to combine their efforts to ascertain the true significance of the finds. Therefore, at the beginning of this report, they wish to express their sincere thanks to Major MALLINCKRODT for his co-operation with the official services, witnessing to his true scientific interest and the seriousness of his archaeological studies.

The finding-place (fig. 1, 2) is situated near Hattemerbroek, E. of Wezep, somewhat south of the railway between Amersfoort and Zwolle, and close to the cart-road known as "Keizersweg".

In the meantime identical finds have been made by Major MALLINCKRODT in a loam-pit on the artillery-range of Oldebroek, while the same enthusiastic investigator collected some similar artefacts a short time ago in the clay dredged near Vollenhove and used for the dike being constructed round

the new North-East-Polder in the former "Zuiderzee". In the following, however, we shall confine ourselves to the first-mentioned finding-place near Wezep.

It should be mentioned that the two last-named authors are responsible for the geological, the first-named for the archaeological part.

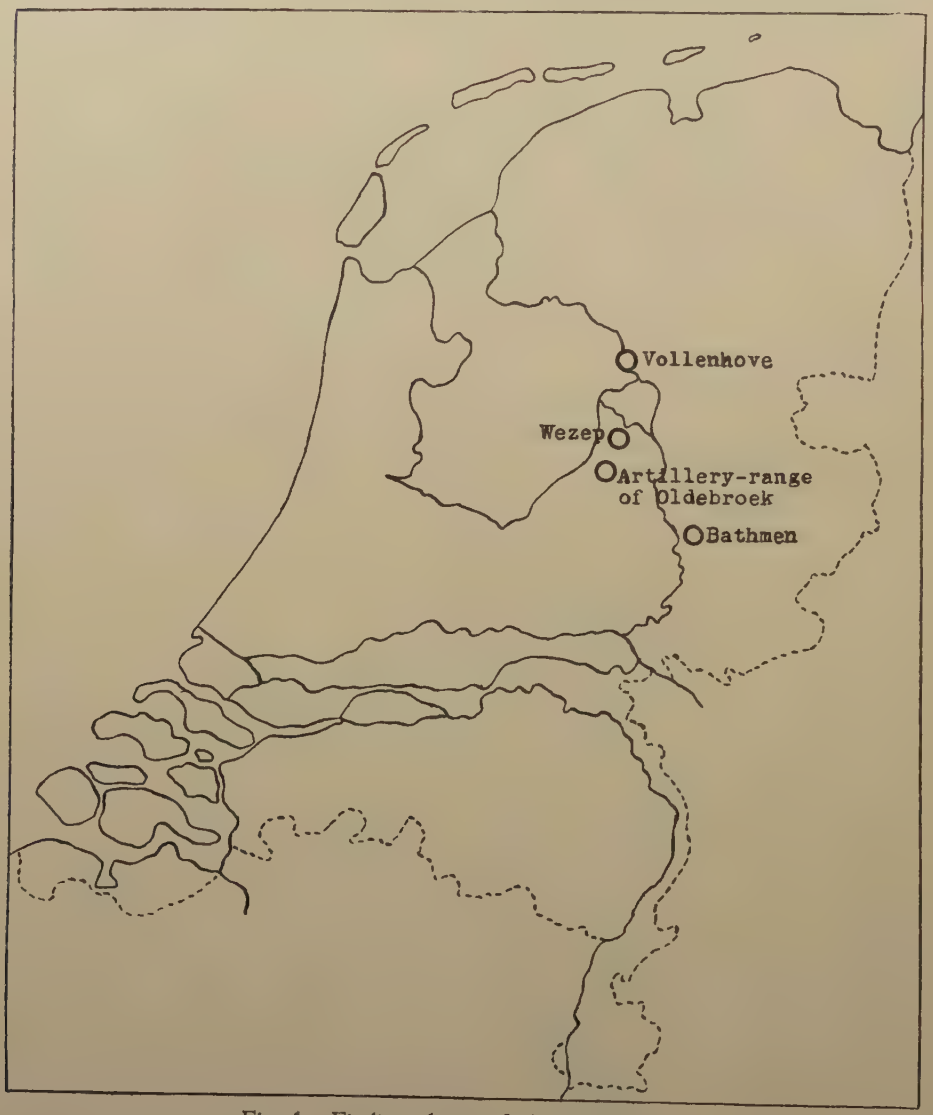

Fig. 1. Finding-places of the artefacts.

I. *Geological part.*

In order to make a geological computation of the age of the worked flints found in the loam-pit near Wezep (fig. 1, 2), the pit was dug out to 3.70 m below the surface of the soil. This revealed the profile shown on the plate and on the drawing (fig. 3) described below.

From the bottom of the pit to 2.60 m below the surface of the soil a complex of coarse sand and gravel was seen, whose original, almost horizontal strati-

Fig. 2. ◉ Finding-place of the artefacts near Wezep. Scale in meters.

fication apparently had been disturbed afterwards. As the gravel contained only southerly erratica it was natural to conclude that this complex (stratum *a* on photos and drawing) was part of the Higher Terrace pushed up by the inland-ice, marked on the geological map of the Netherlands 1 : 50.000 by the symbol II 2 (TESCH, 1927). In agreement with the accepted opinion in our country, we regard this Terrace as having been deposited in the beginning of the Saale-Riss Ice-age, when the Rhine and the Maas built up a delta in this region, which subsequently — but as we shall see at a considerably later period — was partially covered and ridged up by the inland-ice.

On the Higher Terrace, in this profile, reposed a system of strata, marked in the drawing *b 1—5*. This group consisted principally of liver-brown clayish sand (*b1, b3* and *b5*) in which numerous lenses of fine yellow sand were interstratified (e.g. *b2* and *b4*), varying in thickness (maximum 5 cm) and extent. In the for the rest finely granulated sand, many northern erratic boulders were found, some of which had a diameter of a few decimeters. Thus, in the hole visible in *b* on the lower photo, a large flint has lain.

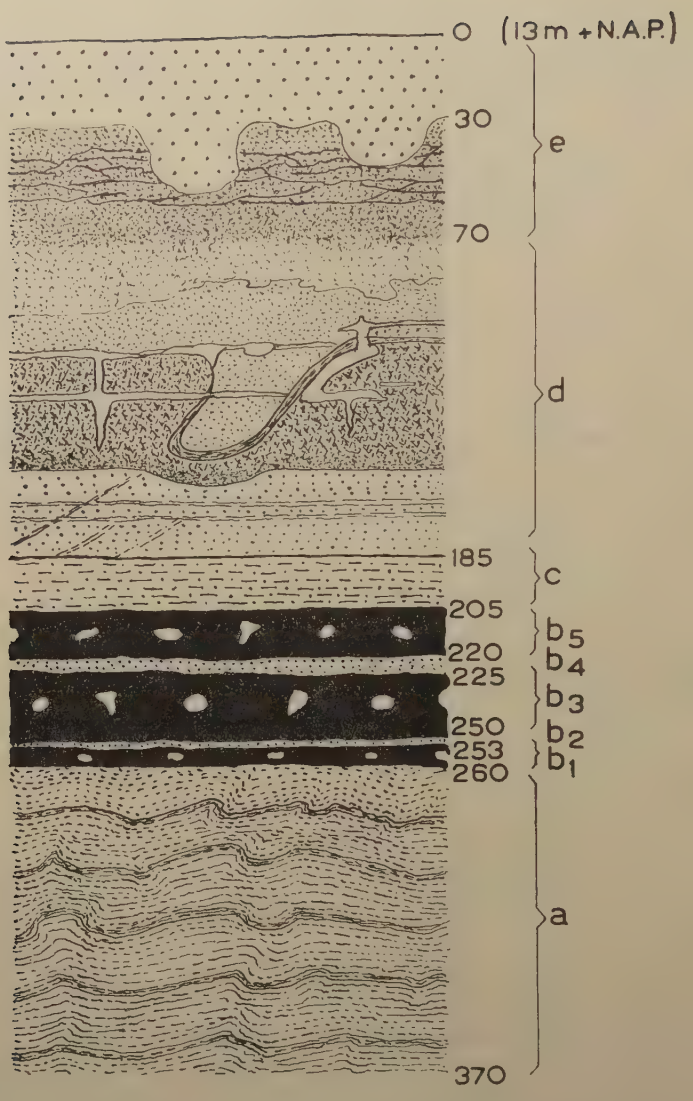

Fig. 3. The section near Wezep [1]).
Scale in centimeters. N.A.P. = Ordnance Datum (Amsterdam).

The artefacts are found exclusively in the strata *b1*, *b3* and *b5*.

Although the material of group *b* showed a certain stratification, it seems to us, regarding the dimensions of the erratic boulders, that it should be considered rather as a modified boulder-clay than as a fluvioglacial deposit. We were considerably strengthened in this opinion by an inspection of the second finding-place of similar artefacts, discovered by Major MALLINCKRODT 10 km S.W. of Wezep on the artillery-range near Oldebroek, where we found an analogous profile, but with this difference, that

[1]) The drawing of this profile we owe to the skilful hand of Prof. Dr. B. G. ESCHER, of Leiden, who with great courtesy accompanied us on one of our visits to the finding-place near Wezep.

The loam-pit near Wezep. General view. August, 1938. F. F.

The loam-pit near Wezep. Detail. August, 1938. F. F.

the stratum *b* from Wezep here is represented by a grey, brown-spotted plastic clay with northern erratic blocks, indubitably boulder-clay. The clay dredged at Vollenhove which contains implements of the same kind, as Major MALLINCKRODT informed us, is also a real boulder-clay.

Stratum *c* of the profile at Wezep had a thickness of 20 cm and consisted of finely stratified clayish sand with various small stones of northern origin. We incline to regard this bed as a fluvioglacial deposit, formed in the melting period of the Saale-Riss inland-ice, especially as its position and structure correspond with the description given by TESCH of such sediments (TESCH, 1924). Above *c* lay a complex of sand (*d*), again containing northern erratic boulders, which was comparatively fine and clayish in the lower part, coarse-grained in the upper. The complex included a small bed of greyish-white sandy clay, entirely free from stones. Stratum *d* showed very clear signs of kryoturbate disturbance, consisting chiefly of pockets of the greyish-white clay and the coarse sand protruding into the lower finer deposits.

Kryoturbate disturbances were first described in the Netherlands in connection with what the excavations for the Twenthe-canals brought to light (FLORSCHÜTZ, 1934; EDELMAN, FLORSCHÜTZ and JESWIET, 1936) and were later signaled from various parts of our country (EDELMAN, 1938). In the cases where we studied these disturbances more narrowly, it was shown that they dated from the end of the Late-glacial or the beginning of the Holocene times (FLORSCHÜTZ and VAN DER VLERK, 1937 and 1938), although the deposits in which they appear may be of greater age. In the profile in question, stratum *d* which was influenced by the kryoturbate movement, consequently cannot be younger than early-Holocene, but might very well have been deposited in the end of the Saale-Riss glacial, the Eem interglacial or the Weichsel-Würm glacial times.

The same may be said of stratum *e*, consisting of coarse sand, in the upper part especially mixed with gravel, showing a distinct podsolation. The complete absence of any visible connection with the kryoturbate phenomena, however, forces us to consider the possibility that layer *e* was formed entirely in the Holocene. The absence of animal and vegetable fossils prevents further precise determination of the geological age of the complex *c*—*e*, which, nevertheless, is immaterial in determining the stratigraphic position of the artefact-layer *b*.

On the basis of these geological data, we can now make an attempt to estimate when and where the men who used these artefacts, have lived.

The first thing we may assume is that it must have been before the formation of stratum *b* and at a more or less considerable distance N. or N.E. from the finding-place. It cannot be supposed that this people inhabited the neighbourhood of Wezep, when it was covered by ice. They must have lived before that period, in all probability after the formation of the Higher Terrace, as in its deposits there is a complete absence of flint-implements.

Between the formation of the Higher Terrace and the deposit of the boulder-clay, a considerable time must have intervened. This is demonstrated by the fact that after the formation of the Higher Terrace the large rivers had the opportunity to grind out deep valleys in the delta before the inland-ice forced them to change their course, so that the boulder-clay or the remnants of it in the IJssel-valley near Deventer and the Rhine-valley near Zevenaar, for instance, came to be situated respectively 100 and 50 m below the present surface.

STEENHUIS recently discovered that in this erosion-period of the Saale-Riss Ice-age local accumulation took place before the deposit of the glacial drifts.

We borrow the following from his preliminary publication (STEENHUIS, 1937) and written communications which he has kindly permitted us to make use of.

In two districts, namely in Limburg and North Brabant, within the territory of the Central Graben, and in Gelderland and Overijssel in the neighbourhood of the old IJssel and the Regge, principally, but also in various other places situated more to the north in Overijssel and Friesland, STEENHUIS found on the Higher Terrace layers of peat, humic clay and fine sand. These in their turn, were covered by sediments from a later phase of the Saale-Riss glacial time, namely in the south of our country by Middle Terrace-deposits and in Gelderland, Overijssel and Friesland by pre-morainal-fluvioglacial material, ground-moraine and postmorainal-fluvio-glacial sediments.

The preliminary botanical examination of samples of these interstratified sediments — which the Director of the "Rijksbureau voor Drinkwatervoor-ziening" at the Hague kindly put at our disposal — throws some light upon the climatic conditions at the time of their formation. In the macroscopic examination, relics were found of a temperate-thermophilous water- and bog-flora (*Alisma, Alnus, Batrachium, Carex, Ceratophyllum, Chara, Comarum, Euphorbia, Hippuris, Limnanthemum, Lycopus, Menyanthes, Myriophyllum, Nuphar, Potamogeton, Ranunculus, Rubus, Salvinia, Scirpus, Sparganium, Stellaria, Viburnum, Zannichellia*), while the pollen-analysis yielded spectra with *Abies, Alnus, Carpinus, Picea, Pinus* and *Corylus*, indicating the presence of temperate-continental forests in the area. Taken as a whole, this vegetation does not differ from that during the formation of the lacustrine sediments in the Eemian Sea-epoch near Zwolle and Zutphen. Both are distinguished from the flora of the penultimate interglacial period (Mindel-Riss) by the absence of *Azolla filiculoides* Lam., which, to a certain extent, acts as an index fossil for the deposits from that interglacial time (horizon of Neede) (FLORSCHÜTZ, 1938).

Having arrived at the conclusion that the makers of the implements must have lived in an episode between the formation of the Higher Terrace and

the deposit of the boulder-clay, we can go further and estimate the time more closely.

We must bear in mind that it was only glacial drift-material and not the Higher Terrace that could provide them with the flints from which they knapped their implements. The glacier or, as is not unlikely, the melting-water which preceded the glacier must have been in the vicinity of the dwelling-place of these men and brought the flints with it. This is very plausible, as we must suppose that the inland-ice entering our country sent out glacial streams and subsequently tongues of ice, first of all into the existing valleys, such as that of the IJssel between Deventer and Zwolle. They further excavated these depressions, that thereupon became completely filled up by the glacier which ascended the valley-slopes, pushed up the Higher Terrace and finally covered it. We must also consider that the advance of the continental ice did not proceed at a steady pace. Undoubtedly there will have been periods in which the ice retreated to some extent, in other words, the advance must have been of an oscillating nature. A temporal, partial filling of the IJssel-valley, either by outwash-deposits or by the ice itself, would have been sufficient to supply the desired flints.

If our conclusion is correct that the implements belong to the advanced Saale-Riss glacial age, it follows, according to the radiation-curve of MILANKOVITSCH and the ice-age-curve constructed by SOERGEL, that those who made the artefacts must have lived between 200.000 and 180.000 years before the commencement of our era. (KÖPPEN and WEGENER, 1924; SOERGEL, 1937).

The hypothesis that the men who knapped the flints found at Wezep, were living in the Saale-Riss glacial time, is also supported by the fact that recently at several places in Germany an analogous culture has been found, which is placed by German geologists and archaeologists in the same age (WOLDSTEDT, 1935; GRAHMANN, 1938; ZOTZ, 1938).

The climatic conditions must not necessarily have been unfavourable in that episode neither; on the contrary, pollen-analysis of samples of the so-called "potklei" from the province of Groningen, clay which is considered as having been deposited in lakes of melting-water from the advancing glacier, have so far yielded spectra with *Abies, Alnus, Betula, Picea, Pinus, Quercus, Salix* and *Corylus*, so that the continental ice apparently penetrated through a temperate-continental forest and not through a barren tundra. Up to the present, in fact, nothing has shown that even at the maximum extension of the glacier, it was extremely cold in the Netherlands south of the ice-border. There are found no relics in our country of a tundra-fauna (VAN DER VLERK, 1938) and -flora, nor kryoturbate phenomena dating from that ice-age, while from other sources arguments can be produced against an arctic climate (SCHREUDER, 1936). This is in contrast to the conditions during the Weichsel-Würm glacial period, when the glacier did not pass the Elbe (SCHREUDER, 1936; FLORSCHÜTZ, 1930 and 1936).

Fig. 4. Artefacts from Wezep. Natural size.

Fig. 5. Artefacts from Wezep. Natural size.

Having made in this way an attempt to compute the geological period in which this people lived, we have still to answer the question where these men lived. And just as there is no reason for us to go further back in time than to the phase immediately preceding the glaciation of our country, neither are there any grounds for supposing that their dwelling-place was far removed from the finding-spot of the artefacts, although it must be acknowledged that this cannot be entirely excluded. The fact that most of the implements show no signs of having been conveyed, may it be in the ground-moraine, renders it, however, improbable. The assumption, therefore, that our men lived on the high western border of the contemporary IJssel-valley, not more than some kilometers to the N.E. of Wezep, will meet with few objections. They may have made their tools on the spot, and left a number of the artefacts behind them, when they were driven away by the glacier which rose out of the valley.

In conclusion we may say that the artefacts which are found near Wezep, once belonged to men who lived before the deposit of the boulder-clay, in all probability after the formation and even after the scouring-out of the Higher Terrace, that is to say about ± 190.000 years before our era. Perhaps this people had a settlement on the western side of the depression what we now call the northern part of the IJssel-valley.

II. *Archaeological part.*

Besides numerous waste and half-made pieces, we have found at least near Wezep some 50 flints which are clearly recognizable as human implements. Of course steps have been taken so that in the steady working going on at this place the artefacts that are revealed, shall be collected for the National Archaeological Museum. The present publication is necessarily of a preliminary character, as a larger collection of these flints may perhaps enable us to make sharper distinctions.

The material is fairly good flint, partly covered by the natural rough crust. Many of the artefacts have a white patina ("white implements", fig. 4, except the left one on the third range from above), some with rusty-brown spots. Although we may assume that the tools found were not all made or used on the same place, most of them do not show the smooth surfaces that would indicate their having been carried a long way. Most of the artefacts, however, display a modification of the original sharp edges caused by pressure, but this is to be attributed to the effect of human hands and not to natural forces.

The bulb of percussion is struck off. The striking platform is large, prepared and flat and forms an obtuse angle with the removed side. The cleft apparently was made by strokes with a stone, the finer trimming by pressure at the sides. Thus the forms illustrated in figs. 4 and 5 were evolved: several points beside some awls and scrapers. Many of the artefacts have a strong resemblance to those which have been called Levallois-flakes.

On the basis of this short description of their forms we will now, as far as is possible, place our finds, according to their typological characteristics, in the nowadays accepted schedule of the earliest human civilization.

It can be said immediately that the primitive nature of the implements points to an early palaeolithic culture, while the absence of larger, doubly worked hand-axes excludes them from being assigned to the Chellean or Acheulean.

It is only in recent years that the earlier division of the Lower Palaeolithic into Chellean (Praechellean), Acheulean and Mousterian successively has proved to be unsatisfactory, as these cultures cover a comparatively small area, confined to the middle and south of France and the Pyrenean peninsular.

Besides, small industries have been discovered first in England which are partly contemporary with, and partly even older than the hand-axes-culture. These industries are called in chronological order Clactonean and Levalloisian. It is to this cultural family, found chiefly in north-west Europe, but apparently just as well present in the middle of Germany (Markkleeberg, Vahrholz) (GRAHMANN, 1935 and 1938; WOLDSTEDT, 1935; ZOTZ, 1938), that our pieces bear the most affinity. Our preliminary impression is that the implements of Wezep most resemble those new finds in Germany and those former ones, known under the name of "Pre-mousterian", whilst they may probably be placed in the chronological system constructed by BREUIL (BREUIL, 1932; BREUIL and KOSLOWSKI, 1934), either in the second period of the Clactonean or, more probably, in the earliest Levallois-cultures of the Somme. Since, according to BREUIL, these industries occurred before the close of the Saale-Riss Ice-time, we, following a completely different way, arrive with regard to the age of the artefacts of Wezep at the same results as FLORSCHÜTZ and VAN DER VLERK reached by geological considerations [1]).

Through these discoveries the history of the human inhabitants of our country has again been traced back many more thousands of years.

It is to be hoped that the ever increasing number of amateurs in the Netherlands who devote themselves with so much enthusiasm to the collection of artefacts, may be encouraged by Major MALLINCKRODT's example and the success it met with, to co-operate with the official services in the effort to extend our knowledge of the earliest inhabitants of our country.

Leiden
Velp (G.), September 6th, 1938.

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[1]) In this connection it may be useful to point out that the two artefacts from Bathmen, near Deventer, now in the British Museum in London, are likely to belong to a culture contemporaneous with those of Wezep, Oldebroek and Vollenhove.

At a recent visit to London, where I had the opportunity of studying the descriptive article where the circumstances of the finding of these artefacts are given, I found no ground whatever to doubt their genuineness in any respect.

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Anatomy. — *The External Morphology of the Brain of Notoryctes typhlops.* By A. N. BURKITT. (University of Sydney). (From the Dutch Central Institute for Brain Research, Amsterdam). (Communicated by Prof. C. U. ARIËNS KAPPERS).

(Communicated at the meeting of September 24, 1938.)

The following account is based upon material collected for the Department of Anatomy of the University of Sydney. A number of specimens of Notoryctes in varying conditions of preservation (all now in 10 % Formalin) have been collected chiefly from Ooldea on the East-West Railway line on the South Australian border or from Hermannsburg. I am indebted particularly to Mr. SMART formerly of Ooldea, to Mr. PETERING of Hermannsburg and Mr. LARNACH and Mr. SCHAEFFER of my Department for their help in collecting and preserving these specimens. The brains were in all cases preserved by carefully opening the skull and membranes dorsally on one side of the mid line and then immersing the animal in 10 % formalin. The abdomen was also opened. In most cases the animals were alive when caught, and were killed by chloroform or ether. Most of the drawings are made from a particularly well preserved and dissected specimen, (N. 13).

Much of the description has been confirmed however by an examination of microscopic sections, stained by CRESYL VIOLET, and by the WEIGERT PAL method. These sections include a transverse W. PAL (D. N. 1.), a transverse CRESYL VIOLET series (D. N. 2), a horizontal W. PAL series (D. N. 6), and a sagittal Iron Haematoxylin series of the right half of the brain of N. 13.

I am also indebted to Professor ARIËNS KAPPERS for his kindness in placing all the resources of his Institute at my disposal, for his advice and help, and to Dr. ADDENS for help and assistance, especially with the drawings, while I have discussed many of my problems with Dr. UNA FIELDING of University College, London. For the cutting of the sections, I am indebted to Mr. WISE and Mr. LARNACH of my Department, and to Mr. BROUWER of the Dutch Central Institute for Brain Research.

In regard to the removal of the brain from the skull, owing to the fragility and small size of the brain, it was only after several attempts that practically undamaged specimens were removed. The use of fine eye scissors and scalpels and especially of fine bone forceps for middle ear operations, proved of great value.

The external morphology of the brain has been described in 1895 by

ELLIOT SMITH from imperfect material examined under great difficulties. As his description and figures are published in a journal which is somewhat difficult of access and as they were imperfect and incomplete owing to the nature and preservation of his material, no excuse is needed for the publication of figures drawn from well preserved brains.

In Notoryctes typhlops we have an extremely primitive marsupial brain, in which at the same time the visual centres have either atrophied or possibly been subordinated to other uses, as a result of the atrophy of the eyes. No trace of the optic nerves or chiasma or optic tract could be seen, nor were any traces seen of the 3rd, 4th and 6th pairs of cranial nerves. The rudimentary stalk of the eye vesicle rudiment could be traced to the brain membranes, but no further. The eye rudiment has already been carefully described by Dr. GEORGINA SWEET.

The extremely primitive character of the brain is indicated by the relatively small development of the neopallium, the piriform lobe (or archipallium) forming nearly two thirds of the lateral surface of each cerebral hemisphere, while the olfactory bulbs and tubercula olfactoria are enormous.

Further several other remarkable features are to be seen in the brain.

The peculiar telescoping which seems to have occured in the longitudinal axis of the brain results in the mid brain axis and the aqueductus cerebri lying almost vertical and almost at right angles to the axis of the medulla oblongata and the fourth ventricle (Fig. 2). As a result of this angulation the interpeduncular fossa forms a deep vertical slit, with a sharp pontine edge formed by a narrow ribbon like brachium pontis which is retrotrigeminal and from each lateral angle of the pons the enormous Vth cranial nerve projects forwards.

The peculiar shape of the 3rd ventricle and commissura mollis seen in Figure 2, is probably associated with this antero-posterior compression of the brain stem. Further the remarkable diverticula developed in the IVth ventricle as a result of the development of large lateral nuclear masses which fuse in the middle line, are, I imagine, quite unique.

These are however best seen in microscopic sections.

For descriptions of the brains of closely allied animals, we are especially indebted to Herrick and Obenchain for their descriptions of Orolestes inca and Caenolestes obscurus, and to Y. T. LOO (YÜ TAO LOO) and VORIS, for their descriptions of the forebrain and hindbrain of the opossum, Didelphis Virginiana, while of course the work of ELLIOT SMITH and his students forms the background for much of our work upon the morphology of the mammalian brain.

Of special interest is LE GROS CLARK's recent study of the brain of the Insectivora, because it contains detailed figures and a description of the brain of Chrysochloris hottentotta. (Cape Golden Mole). The resemblances which this brain shows superficially to that of Notoryctes, are remarkable and would almost certainly seem to indicate as Leche remarked in

an earlier paper „dass dieser Fall, bei dem die Annahme eines näheren genetischen Zusammenhanges ganz ausgeschlossen ist, als ein besonders lehrreiches Beispiel, bis zu welchen überraschenden Grade verschiedene Ausgangsformen unter analogen Bedingungen sich einander nähern können, aufgestellt zu werden verdient." Comparison in detail shows the most remarkable similarity to occur in the form of the cerebral hemispheres when viewed either dorsally or laterally. Actually the olfactory bulbs and tubercula olfactoria are proportionately somewhat smaller in Chrysochloris (Cf. LECHE, p. 587). The cerebellum is also extremely flattened and leaflike, but not to the same degree as in Notoryctes. The antero-posterior compression, and resultant angulation of the brain stem axis does not appear to have been quite so marked in Chrysochloris, and the relations of the corpora quadrigemina are somewhat different, as, they lie more vertically in Notoryctes, the superior or anterior colliculus being moulded by the cerebellum and fascia dentata, while the inferior colliculi bulge backwards into the anterior lobe of the cerebellum, and are completely hidden by the cerebellum. The interpeduncular fossa appears to lie more vertically and to be deeper in Notoryctes.

Proceeding now to a detailed description of the brain of Notoryctes, I shall describe the various aspects of the brain, more particularly in relation to the figures.

The Lateral Aspect of the Brain.

Examining the lateral aspect of the brain (Fig. 1), the following remarks may be added to ELLIOT SMITH's original description. First of all, although a distinct rhinal (ectorhinal) fissure is not visible, nevertheless in fresh formalin preserved brains, a distinct line of demarcation, (confirmed by histological examination) indicates this fissure, and is shown in Figures 1 and 3. It will be seen that it lies very high up on the cerebral hemisphere, and macroscopically appears as a broad strip of transitional cortex, rather than as a sharply defined furrow. Histologically however the transition between piriform cortex and neopallium is quite sharp and this broad band of transitional cortex really belongs to piriform cortex and its apparent differentiation from the lower part of the piriform cortex may be due to underlying structures. The rhinal fissure was also indicated by two blood vessels (?veins) one running forward from the transverse sinus and one backward from the transverse olfactory sinus.

In other respects the proportions of neopallium to archipallium (piriform lobe) and palaeopallium approximate most closely to Orolestes inca and Caenolestes obscurus, except that the olfactory bulb and tuberculum olfactorium attain an even greater relative size than in either of the above animals. Further the fissura orbitalis of these forms is absent in Notoryctes, as might be expected from the rudimentary condition of the orbit. It is of considerable interest to note that the amygdaloid nuclei (archi-

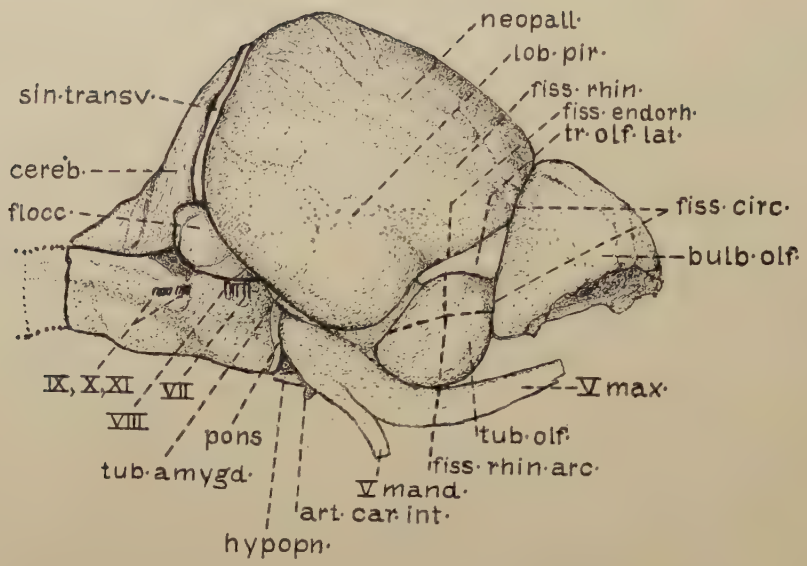

Fig. 1. Norma lateralis. (N. 13 Anatomy Dept., University of Sydney) Magnification × 5. The right side was drawn, contrary to general custom, because the flocculus was intact on this side alone. The hemisphere was 8.4 mm long, the Olfactory bulb 4.3 mm long, the flocculus 3 mm long (approx.) and the lateral Olfactory tract 3 mm long (i.e. the medullated white band which is visible macroscopically and which apparently ends abruptly posteriorly).

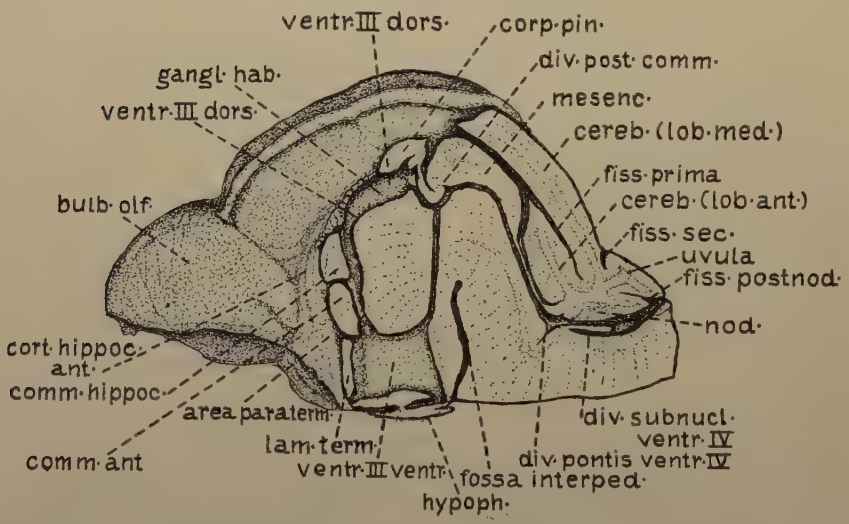

Fig. 2. Median Sagittal Section. × 5. The floor of the third ventricle is shown as being of considerable thickness, because the section was slightly to one side of the middle line. Actually in the midline, the floor is reduced to a microscopic epithelial floor plate. The Lamina terminalis on the other hand is definitely very thick in section, and is quite correctly drawn. The communicating ducts of the third ventricle are also actually somewhat smaller, and the posterior one is in places microscopic.

striatum) appear to equal or be even larger in size than the neostriatal (caudate-putamen) nuclei, a reptilian feature. In other respects it will be noted that the formalin hardened brain differs a little in form from that figured by ELLIOT SMITH (E. S. Fig. 1) (cf. natiform eminence below) and that the white colour of the external olfactory radiation appears to end abruptly at an indentation (Pseudosylvian?) of the piriform lobe.

The great proportional size of the olfactory bulb and tuberculum olfactorium are very evident, while the nucleus of the external olfactory radiation and the nucleus amygdaloideus can both be seen from the lateral aspect. It will be seen that the piriform lobe shows some differentiation into an upper transverse band already described, and a marked natiform eminence as described by ELLIOT SMITH, and areas anterior and posterior to this eminence, The area posterior to the main projection of the natiform eminence is of course simply a backward extension of this eminence as is clearly shown in figure 3, but I mention it, as it is not clearly indicated in ELLIOT SMITH's figure, probably because of shrinkage of the brain and it will be seen that my lateral view differs chiefly from ELLIOT SMITH's in the great size of this cap like backward extension of the natiform eminence, so that the hemisphere appears more irregularly four sided than triangular in my specimens.

The Basal Surface.

I have been unable to confirm ELLIOT SMITH's description (cf. his figure 2) showing the lateral olfactory radiation extending round the posterior aspect of the tuberculum olfactorium, and ending in a small medial tubercle.

In my material it seems to end somewhat more laterally in a slight swelling. This was confirmed in CRESYL VIOLET sections in which the tract appears to end in relation to a marked nucleus, whose cells stain somewhat more deeply than those of the surrounding nuclei, i.e. nucleus of the lateral olfactory tract. The amygdaloid nuclear mass forms a distinct tubercle and is defined by a shallow but distinct groove from the piriform lobe laterally. The amygdaloid tubercle may also be clearly seen in Figure 5 and is just visible in Figure 1. The outstanding features of the ventral aspect are of course the huge olfactory bulb and olfactory tubercle, the large natiform tubercle of the piriform lobe, and the enormous Vth cranial nerve. (This nerve was transversely grooved on its ventral aspect on both sides of the brain.) The pituitary body, a flattened slipper like structure, is shown with the internal carotid artery on either side. It is 2.6 mm. long, and 2 mm. broad approximately.

Owing to the presence of the enormous Vth cranial nerve, the diagonal band of Broca is not shown in the figure, but a definite thread like white structure could be traced running from the paraterminal region, or really from the lower end of the anterior hippocampal cortex, skirting the nuclei

of the floor of the third ventricle and reaching the lower end of the posterior hippocampal cortex or fascia dentata on the posteromedial border of the amygdaloid tubercle. This was confirmed microscopically.

Detailed comparison can perhaps most easily be made with the extremely accurate and detailed figure of Loo (Fig. 1. Pt. 1, 1930) of Didelphys virginiana.

The Medial Section.

Several unusual features may be seen, some of which have already been briefly mentioned. The third ventricle has a squarish slit like ventral part (Vent. III. v) from which the hypophyseal recess projects ventrally and posteriorly. Dorsally two small tubular cavities or ducts connect with the dorsal part of the third ventricle (Vent. III. d), in front of and behind the commissura grisea thalami (or massa intermedia). The posterior con-

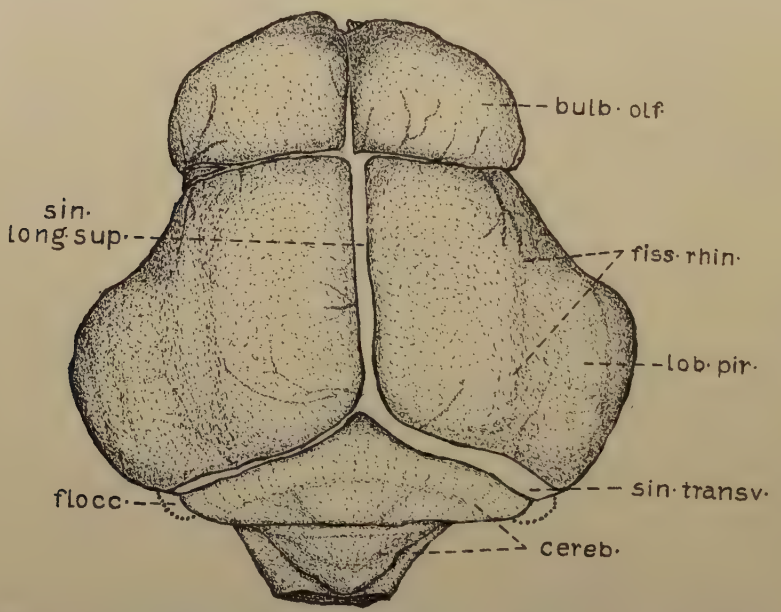

Fig. 3. Norma dorsalis. × 5. The approximate position of the flocculi is indicated by dotted lines.

nection is particularly small and quite thread like, so much so that at one time I thought it was absent, and only more careful examination revealed it in sections. There are also two recesses anteriorly, one immediately below the anterior commissure, and one ventro-anterior in position. The floor of the third ventricle is really much thinner in the midline, the section which was drawn having been slightly to one side of the midline. The dorsal part of the third ventricle has an anterior recess above the commissura hippocampi, and the ganglion habenulae makes a very marked and elongated (antero-posterior) ridge projecting into the ventricle; in

regard to this supracommissural recess however, I have been unable to find that it presents any special characters which would indicate that

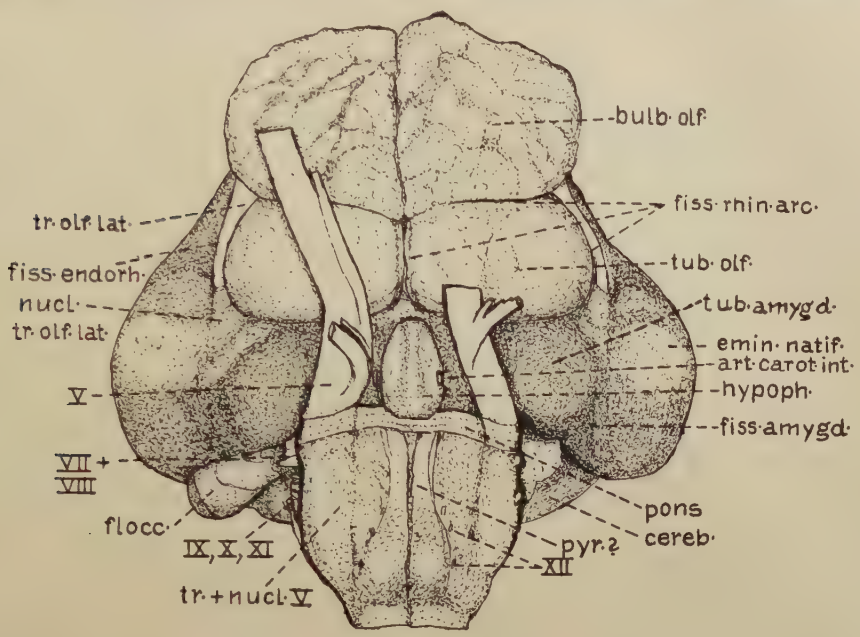

Fig. 4. Norma basalis. × 5. The Vth cranial nerves have been left in situ to show their enormous size. The left flocculus was damaged during removal of the brain. The medial part of the Fissura rhinalis arcuata is a little over-emphasized, but is definite, and the resultant lipping of the medial borders of the tubercula olfactoria is due to the downward projection of septal or paraterminal nuclei and cortex hippocampi anterior. Measurements. Total length of the brain from the anterior end of the bulbus olfactorius to the medullary depression (spinal flexure) 15 mm. Greatest breadth of the brain 15 mm. Length of Bulbus olfactorius 5 mm. Length of Tuberc. olfact. 3 mm. Breadth of Bulbus olfactorius 5 mm. Breadth of Tuberc. olfact. 5 mm. Length of Hypophysis circa 2.6 mm. Breadth of Hypophysis circa 2 mm.

it is a remnant of a definite diverticulum or paraphysis, such as has been figured by WERKMAN in embryos of Vesperugo noctula, etc. Dorsally the pineal recess seems to be double, but in my sagittal sections this does not appear to be the case, there being a recess anterior to the pineal body, bounded anteriorly by choroid plexus and posteriorly by pineal body, and a pineal recess, and no post pineal recess, such as seemed to be present in the macroscopic specimen. This latter may have been an artefact.

The postcommissural dorsal blind diverticulum at the anterior end of the aqueductus cerebri is a very marked duct like structure in microscopic sections. Next come the pontine flexure diverticulum, and then the remarkable intraventricular diverticulum on the anterior part of the floor of the fourth ventricle.

The deep interpeduncular fossa, anterior to the pons, is clearly seen and the marked flexure of the whole brain stem is well shown. The

aqueductus cerebri and the ducts of the third ventricle anterior and posterior to the massa intermedia are quite microscopic in places and are often actually much smaller than shown in the figure. The simplicity of the cerebellum is clearly shown, and this has already been described by ELLIOT SMITH.

From the point of view of brain mechanics, however, the cerebellum presents several interesting problems. I am inclined to think that the relatively enormous flattened leaf like lobus medius is so shaped in correlation with the large flattened supraoccipital bone surface, which is present for the attachment of powerful neck muscles, and this latter feature is again in correspondence with the torpedo like shape of the head, so well adapted for burrowing in sand (cf. LECHE). As a result of the skull moulding, the whole cerebellum is very compressed anteroposteriorly, and the posterior corpora quadrigemina form deep depressions on the anterior wall of the lobus anticus. One feature of interest is not mentioned by ELLIOT SMITH, and that is the fact that the lobus anticus has two side wings or lappets which fit into deep pockets on the anterior wall of the lobus medius, so that in horizontal section we have the lobus anticus appearing as a very flattened oval with the two pointed ends almost surrounded by the lobus medius, which must have grown round these lateral wings of the lobus anticus. The remaining details of the morphology of the cerebellum have been very carefully and fully described by ELLIOT SMITH. Also the paraterminal region has been clearly described by him, and all I have to add is that the faint vertical furrow upon its surface has probably some morphological value, though it is also apparently related to anterior cerebral vessels. It probably separates, approximately, a narrow tapering strip of anterior hippocampal cortex from a broader posterior paraterminal area, with the septal nuclei underlying it. With a high power binocular dissecting microscope I was able to see white strands of the fornix precommissuralis shining through the surface cortex and running almost vertically in the paraterminal region. The fissura circularis is apparently quite deep on the medial aspect from an examination of microscopic sections but appears merely as a shallow groove macroscopically and under high powers of a binocular dissecting microscope.

The full extent of the midline surface of each cerebral hemisphere is clearly shown by a comparison of figures 2 and 5, and it will be at once apparent how the postero-ventral part of each hemisphere is pushed laterally and forms a cap over the diencephalon and upper part of the midbrain, as already clearly described by ELLIOT SMITH (p. 179).

The Dorsal Aspect.

Figure 3 gives a clear representation of this aspect and requires few remarks. The fissura rhinalis, which seems to be faintly indicated, may actually not be a very accurate guide to the extent of the neopallium,

as my microscopical material seems to show that the neopallial cortex may be even narrower than indicated macroscopically. Further in none of my

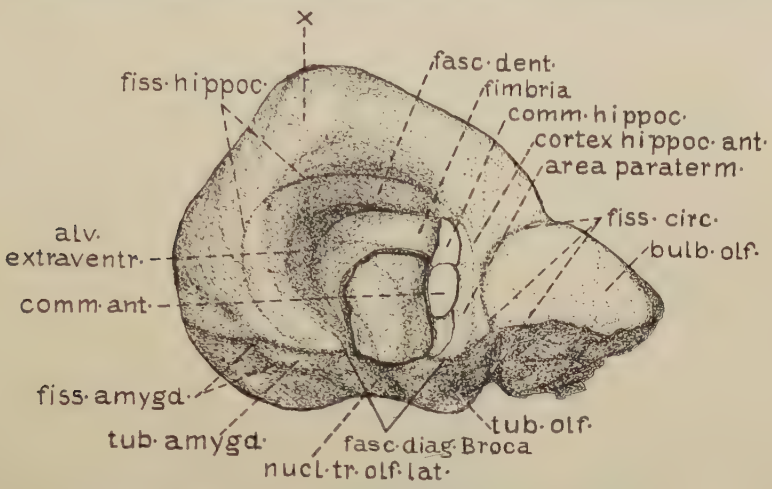

Fig. 5. Median aspect, (× 5) of the left cerebral hemisphere with the diencephalon and remainder of the brain removed. The cut surface of the diencephalon is shown. This cut surface was not flat but was formed by two planes (knife cuts) forming almost a right angle, the anterior plane being transverse, immediately behind the commissures, and the posterior plane parasagittal.

formalin hardened specimens was much, if any, of the midbrain visible between the cerebellum and the hemispheres, as described by ELLIOT SMITH. This is, of course, a minor matter, and may have been due to his material being fixed in alcohol. The approximate extent of the lobi flocculi which were destroyed in the brain figured, is shown by dotted lines.

The Basis Cranii.

Finally a sketch is given of the basis cranii, after removal of the brain. In this sketch the outstanding feature is the enormous size of the Vth cranial nerves, which in a fresh formalin preserved brain show up as two large sickle shaped creamy white structures on either side of the hypophysis. From before backwards we have the large jelly like mass of the nasal cavities and sinuses, the broad triangular cribriform plate area and behind this an oblique slope of bone on either side, through which the mucous membrane of the nasal cavities shine. As already mentioned, the large Vth nerves can be well seen on either side of the pituitary fossa bordered medially by blood clot in the cavernous sinus. No trace of the II, III, IV and VI cranial nerves could be seen in this region with high powered dissecting binoculars (× 20 circa). Two structures which were probably two internal carotid arteries could be seen piercing the dura mater and these are figured. The blood clot, presumably in the large cavernous sinus, extends forwards in the midline to the cribriform plate.

In the posterior cranial fossa the 7th and 8th nerves could be seen and are relatively small and immediately anterior to and below the large

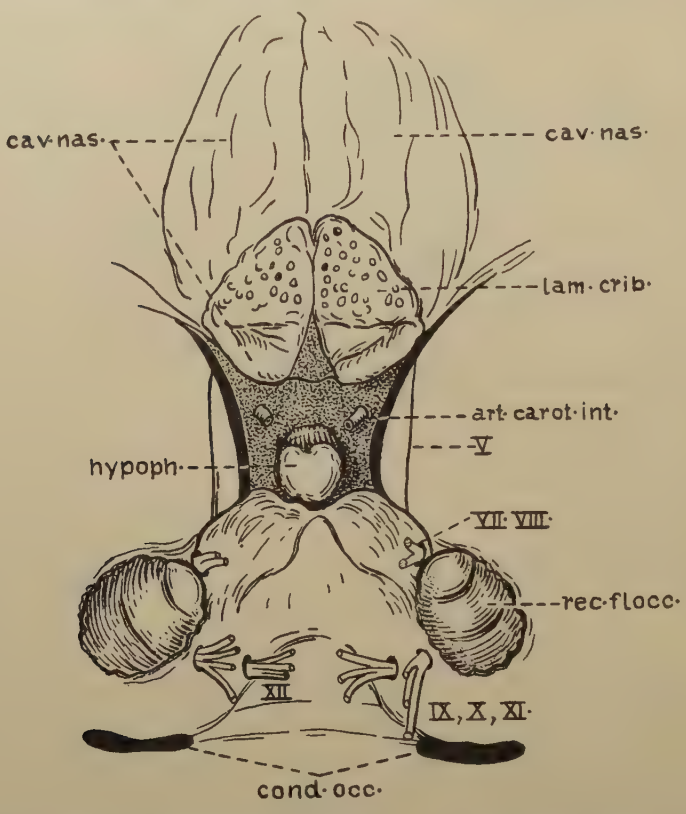

Fig. 6. Sketch (approximately to scale × 4) of the midline region of the dorsal aspect of the basis cranii (interior of the skull), after removal of the roof of the skull and brain. To show the cranial nerves passing to their foramina of exit. The hypophysis was surrounded by blood clot and the roof of the floccular recess has been broken away on either side. The nasal cavities are shown very roughly in section, anterior to the cribriform plate, while behind this cribriform area is a smooth and very thin plate of bone on either side, through which the mucous membrane of the hindmost part of each nasal cavity shines.

floccular recess. The divisions of the 8th nerve could not be observed with certainty.

The 12th nerve was well developed with at least 3—4 rootlets, and the 9th, 10th and 11th nerves could be seen, but only the posterior spinal portion of the spinal accessory could be distinguished with certainty and this is figured. The floccular recess appeared to be ridged in correspondence with the floccular subdivisions. These recesses had very narrouw necks which made it extremely difficult to remove the flocculi intact.

The account would have been more complete, if figures could have been given of the appearance of the dorsal aspect of the mid brain and

of the interpeduncular fossa. Owing to paucity of material and fear of damaging the specimens before sectioning, this was not possible. I can however make a few remarks upon these regions. On opening up the interpeduncular fossa in one of my earlier specimens, a single median corpus mamillare could be seen anteriorly and on the posterior wall two rod like lateral fibre masses, the edges of the pedunculi cerebri with a median tubercle, the nucleus interpeduncularis. As regards the roof of the midbrain, the most outstanding feature is the great development of the posterior corpora quadrigemina, which form two large rounded masses which project backwards and lie in definite hollows which they form in the anterior wall of the cerebellum on either side of the midline.

The Spinal Cord.

The brain and spinal cord which form the WEIGERT PAL Series D.N.I., were removed as a whole, the spinal cord was 7 cM long with definite brachial and lumbo-sacral swellings. The brachial swelling in consonance with the reduction of the neck region, seems almost continuous with the brain stem, except for a narrow constriction at the flexure which occurs at the foramen magnum. The spinal cord joined the brain stem at a slight angle, thus forming a spinal flexure.

Summary.

This paper gives drawings and a description of the external morphology of the brain of Notoryctes typhlops, and is an amplification of the classic description of ELLIOT SMITH in 1895.

Notoryctes has probably the most primitive and relatively unspecialised mammalian brain known, with certain peculiar features which further add to its importance and interest. The cerebral hemispheres have an enormous piriform lobe, and a very small neopallium; the olfactory bulbs, tubercula olfactoria, and the amygdaloid tubercles are very large. There is no trace of the optic chiasm. The third, fourth and sixth cranial nerves are absent.

The fifth cranial nerve is strongly developed, the ophthalmic division however being very small and atrophied.

The third ventricle, the aqueductus cerebri, and the fourth ventricle present a number of peculiar features, partly associated with the peculiar antero-posterior compression of the brain stem. The midbrain-pontine angle is very marked, so that the midbrain and hindbrain lie almost at right angles to one another. The fourth ventricle presents a most unusual subnuclear transversely flattened pouch or diverticulum on the anterior part of its floor.

The cerebellum is extremely simple and has also probably been modified in form by the skull.

Comparison with Chrysochloris, the Eutherian mammal most closely resembling Notoryctes superficially, gives us a remarkable example of the

phenomenon of functional convergence in the form of the brain as a whole. The differences have been in part outlined and also present some interesting problems e.g. the different orientation of the tectum mesencephali in the two forms, and the differences in the details of cerebellar morphology.

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ABBREVIATIONS USED IN THE FIGURES.

Alv. extravent	Extraventricular alveus.
Area paraterm.	Area paraterminalis (septalis).
Art. Carot. Int.	Internal Carotid Artery.
Bulb. Olf.	Bulbus olfactorius.
Cereb.	Cerebellum.
Cereb. (lob. ant.)	Cerebellum. Lobus anterior.
Cereb. (lob. med.)	Cerebellum. Lobus medius.
Com. ant.	Commissura anterior (or ventralis).

Com. hippoc.	Commissura hippocampi (or dorsalis).
Cond. occ.	Occipital condyles.
Corp. pin.	Corpus pineale.
Cort. hippoc. ant.	Anterior or precommissural hippocampal cortex or fascia dentata anterior.
Div. pontis Ventr. IV	Diverticulum pontis of the IVth Ventricle.
Div. postcomm.	Diverticulum postcommissurale of the aqueductus cerebri.
Div. subnucl. Ventr. IV	Subnuclear diverticulum of the IVth Ventricle.
Emin. natif.	Eminentia natiformis of the lobus piriformis.
Fasc. dent.	Fascia dentata or gyrus dentatus.
Fasc. diag. Broca.	The two ends of the diagonal band of Broca.
Fimbria.	Fimbria.
Fiss. amygd.	Fissura amygdaloidea.
Fiss. circ.	Fissura circularis.
Fiss. endorh.	Fissura endorhinalis.
Fiss. hippoc.	Fissura hippocampi.
Fiss. postnod.	Fissura postnodularis cerebelli.
Fiss. prima.	Fissura prima cerebelli.
Fiss. rhin.	Fissura rhinalis.
Fiss. rhin. arc.	Fissura rhinalis arcuata.
Fiss. sec.	Fissura secunda cerebelli.
Floc.	Lobus flocculi cerebelli.
Fossa interped.	Fossa interpeduncularis.
Gangl. hab.	Ganglion habenulae.
Hypoph.	Hypophysis.
Lam. crib.	Lamina cribriformis.
Lam. term.	Lamina terminalis.
Lob. pir.	Lobus piriformis.
Mesenceph.	Roof of the mesencephalon.
Neopall.	Neopallium.
Nod.	Nodulus cerebelli.
Nucl. Tr. olf. lat.	Nucleus of the lateral olfactory tract.
Pons.	Brachium pontis.
Pyr ?	? Pyramidal tract.
Rec. flocc.	Recessus floccularis.
Sin. long. sup.	Sinus longitudinalis superior.
Sin. transv.	Sinus transversus.
Tr. and Nucl. V.	Tract and Nucleus spinalis of the Vth cranial nerve.
Tr. olf. lat.	Lateral olfactory tract.
Tub. amygd.	Nuclei amygdaloidei (or amygdaloid tubercle).
Tub. olf.	Tuberculum olfactorium.
Uvula	Uvula cerebelli.
Ventr. III. d.	Dorsal part of the third ventricle.
Ventr. III. v.	Ventral part of the third ventricle.
X.	This marks the limit of the midline surface of the cerebral hemisphere.
V. mand.	Mandibular division of the Vth cranial nerve.
V. max.	Maxillary division of the Vth cranial nerve.